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FOUNDATIONS

von Wright, G. H. On the idea of logical truth. II. Soc. Sci. Fenn. Comment. Phys.-Math. 15, no. 10, 45 pp. (1950).

This paper investigates the decision problem for a special class of formulas of the restricted predicate calculus involving binary relations. The class in question is that composed of formulas obtained from a formula containing only one-place predicates by substituting for a ϕx a predicate of the form $R(x, x)$ or $(\exists y)R(x.y)$. The formulation is complicated and the notation deviates from the usual ones to such an extent that the paper is difficult to follow; moreover, the process is not established in general but only illustrated in a special case. In a recent letter the author states that he realizes these deficiencies and has remedied them in a later publication. For a more detailed review by W. Ackermann see J. Symbolic Logic 16, 147–148 (1951). The basic result appears to be new. [For part I see the same Comment. 14, no. 4 (1948); these Rev. 10, 668.]

H. B. Curry (State College, Pa.).

✶Tarski, Alfred. Some notions and methods on the borderline of algebra and metamathematics. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 705–720. Amer. Math. Soc., Providence, R. I., 1952.

Wird unter einer Algebra (im weitesten Sinne) eine Menge A mit Relationen $R_1, R_2, \cdots$ in A verstanden, dann definiert jede Aussage über $A, R_1, R_2, \cdots$ eine Klasse von Algebren. Die Aussagen, die nur die Ausdrucksmittel der elementaren Prädikatenlogik (d.h. keine gebundenen Prädikatvariablen) sind, definieren die arithmetischen Klassen. Verf. gibt einen Bericht über die hauptsächlich von ihm stammenden Untersuchungen über arithmetische Klassen. Zunächst wird die metamathematische Definition in eine mathematische übersetzt. Der Gödelsche Vollständigkeitssatz für den elementaren Prädikatenkalkül erscheint daher als Kompaktheitssatz für arithmetische Klassen. Es folgt z.B. dass die Klasse aller Körper mit Charakteristik 0 nicht-arithmetisch ist, sondern nur Durchschnitt von abzählbar vielen arithmetischen Klassen. Anderseits liefert der Kompaktheitssatz unmittelbar Existenzbeweise, z.B. für nicht-archimedisch geordnete Körper. Die Vollständigkeit gewisser Axiomensysteme, z.B. des Axiomensystems für reell-abgeschlossene Körper übersetzt sich als arithmetische Äquivalenz aller reell-abgeschlossener Körper, d.h. jede arithmetische Klasse, die einen reell-abgeschlossenen Körper enthält, enthält alle. Jede Aussage, die aus arithmetischen Aussagen zusammengesetzt ist und die z.B. für den reellen Zahlkörper gilt, gilt daher für alle reell-abgeschlossenen Körper.

P. Lorenzen (Bonn).

Sampei, Yoemon. Some remarks concerning identity. J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 109–112 (1950).

Explicit proof that the axiom schemes

$$(A_1) \qquad x=x$$

$$(A_2) \qquad x=y.\supset.\mathfrak{A}(y)\supset\mathfrak{A}(x)$$

are equivalent to the axiom scheme

$$(A) \qquad (\exists y)[x=y.\mathfrak{A}(y)].\equiv.\mathfrak{A}(x).$$

H. B. Curry (State College, Pa.).

✶Feys, Robert. Nature et possibilités de la logique formalisée. Congrès International de Philosophie des Sciences, Paris, 1949, vol. II, Logique, pp. 69–80. Actualités Sci. Ind., no. 1134. Hermann & Cie., Paris, 1951.

This brief "defense and illustration" of formal logic, is in literary form free of logical symbolism. The author claims no new results but surveys with abundant mention of recent writers, the merits and the limitations of symbolic logic, pointing out, for example, that it is no longer part of philosophy, but does include semantics in the sense of Tarski and Carnap. The author proposes and answers some dozen questions that might occur to the novice as to the role of formal logic. There are no explicit bibliographical references.

A. A. Bennett (Providence, R. I.).

✶Meyer, Herman. La negation et la logique. Congrès International de Philosophie des Sciences, Paris, 1949, vol. II, Logique, pp. 91–101. Actualités Sci. Ind., no. 1134. Hermann & Cie., Paris, 1951.

Philosophical considerations on different forms of negation, occurring in different, also non-formalised, systems of logic.

A. Heyting (Amsterdam).

✶Skolem, Th. Some remarks on the foundation of set theory. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 695–704. Amer. Math. Soc., Providence, R. I., 1952.

The first of Skolem's 'remarks' concerns a unifying point of view for alternative set theories, ranging from the extremes of the naive set theory of Dedekind to the simplified theory of types of Russell, the Zermelo and von Neumann theories together with modifications of them falling somewhere in between. He goes on to offer a formulation of the axiom of infinity with the Zermelo set theory (ZST) in the form $\hat{x}Ax\varepsilon D$, where Ax is a sentential function of ZST of the one variable x and D is a symbol for the fundamental domain of objects of ZST. Some suggestions as to how this might be done within a system based upon the simplified theory of types are also given. The third remark is concerned with the Löwenheim-Skolem theorem as applied to ZST. As against what he calls the 'opportunistic' point of view of Bourbaki concerning possible contradiction in the foundations of set theory, and also as against the ramified theory of types and the Hilbert programme of consistency proof, Skolem proposes to build up mathematics 'in a strictly finitary way'. In 1923 [Skr. Norske Vid. Akad. Oslo. I. 1923, no. 6] Skolem gave a formulation of recursive arithmetic without quantifiers. He seems to regard this as the most natural formulation of arithmetic. He believes that much of analysis could be built up on such a basis.

R. M. Martin (Philadelphia, Pa.).

Shepherdson, J. C. Inner models for set theory. I. J. Symbolic Logic **16**, 161–190 (1951).

This is the first of a sequence of three papers dealing with inner models. Given a domain of individuals assumed to satisfy certain axioms, an inner model is one whose domain of individuals is a subset of the original domain of individuals. Such models have been used in relative consistency proofs by various authors, including von Neumann, Mostowski and Gödel. In these papers some general theorems will be proved to show that for a certain system of set theory and a fairly large class of inner models, this method of proving consistency has been exhausted and no essentially new consistency results can be obtained this way. The system of set theory Σ' under consideration is the one used by Gödel in "The Consistency of the Continuum Hypothesis" [Princeton Univ. Press, 1940; these Rev. **2**, 66], but without axiom D (axiom of Fundierung). We shall consider only models for which there exist propositional functions $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ (with or without free variables) such that (i) $\mathfrak{Cls}_m(A) . \equiv . \mathfrak{A}(A)$, (ii) $\mathfrak{M}_m(\mathfrak{A}) . \equiv . \mathfrak{B}(\mathfrak{A})$, (iii) $A \varepsilon_m B . \equiv . \mathfrak{C}(A, B)$, where "$\mathfrak{Cls}_m(A)$", "$\mathfrak{M}_m(\mathfrak{A})$", "$A \varepsilon_m B$" mean respectively "$A$ is a class of the model", "A is a set of the model", "A is a member of B (in the sense of the model)". Such a model is called complete if also the ε-relation of the model is the same as that of the universe, and whenever A_m is a class of the model, then all members of A_m are sets of the model. Every complete model satisfies (i), (ii), (iii). Any statement of the original system in primitive notation can be relativised for the model under consideration in the usual manner. The propositional function $\mathfrak{A}(A, A')$ effects within the given universe an isomorphism of 2 models $\mathfrak{M}$ and $\mathfrak{M}'$ if $\mathfrak{A}(A, A')$ 1) effects a one-to-one correlation of the classes of $\mathfrak{M}$ on the classes of $\mathfrak{M}'$ and 2) maps sets onto sets, and 3) leaves the ε-relation invariant. We then have the following theorem: Any model $\mathfrak{M}$ for the axioms which satisfies (i), (ii) and (iii) and also "$(A)(\sim \mathfrak{Em}(A) \cdot \supset \cdot (\exists y)(y \varepsilon A \cdot (z) \sim (z \varepsilon_m y \cdot z \varepsilon A)))$" is isomorphic to a complete model. Further results of this paper state that for complete models many set theoretical concepts are absolute in the sense that the concepts defined with respect to the model coincide with the corresponding concepts for the original system. Among these are for instance "ε", "$\subseteq$", "$\mathfrak{E}$", "$\{xy\}$", "$\langle xy \rangle$", "$\mathfrak{Un}$", "$A \times B$", "$R \times l$", "$A \cdot B$" and "$\restriction$".

I. L. Novak (Princeton, N. J.).

***Destouches-Février, Paulette. Logique et théories physiques.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. II, Logique, pp. 45–54. Actualités Sci. Ind., no. 1134. Hermann & Cie., Paris, 1951.

Eine "Messung" einer physikalischen Grösse A zur Zeit t liefert als Messresultat $R_t(A)$ eine Zahl. Man kann "Messung" aber auch so definieren, dass das Messresultat ein Intervall $\mathfrak{x}$ ist, oder dass durch Messung nur Aussagen der Form $R_t(A) \subseteq \mathfrak{x}$ zu erhalten sind. Diese heissen experimentelle Aussagen $p(t)$. Nach der Quantentheorie wird jeder exp. Aussage p ein nicht leerer linearer Unterraum M_p einer gewissen Raumes R zugeordnet. Mit Hilfe dieser Unterräume lassen sich daher Operationen für die Aussagen definieren. Dies sind selbstverständlich keine logischen Operationen (ebenso wenig wie z.B. die "Multiplikation" zweier Gleichungen $x=a$ und $x=b$ zu $x=a \cdot b$. eine logische Operation ist), aber da die M_p (einschl. der leeren Menge) einen orthokomplementären, modularen Verband bilden, also einen Booleschen Verband ähneln, spricht man hier von "Quantenlogik". Für solche "Logiken" trifft die These der Verf. trivialerweise zu, dass sie erfahrungsabhängig sind, die eigentliche Logik aber (als Theorie des formalen Operierens überhaupt) ist jedenfalls von solchen "Logiken" unabhängig. Denn ohne sie entstünden solche "Logiken" gar nicht. Von einer exp. Aussage $p_0 = p(t_0)$, d. h. dem zugeordneten M_{p_0} ausgehend, liefert die Quantentheorie für die Zeit t einen Unterraum H_t von R derart, dass für die Wahrscheinlichkeit einer Aussage $q = q(t)$ gilt

$$\text{prob } q = 1 \leftrightarrow M_q \supseteq H_t,$$
$$\text{prob } q = 0 \leftrightarrow M_q \subseteq -H_t.$$

Verf. schlägt vor, "Modalitäten" zu definieren durch

$$\text{Nec } q \leftrightarrow \text{prob } q = 1,$$
$$\text{Pos } q \leftrightarrow \text{prob } q \neq 0.$$

Die Untersuchung einer solchen "Modallogik" wird für die Physik wichtig sein, nach Meinung des Ref. aber auch nur für die Physik.

P. Lorenzen (Bonn).

v. Weizsäcker, C. F. Kontinuität und Möglichkeit. Eine Studie über die Beziehung zwischen den Gegenständen der Mathematik und der Physik. Naturwissenschaften **38**, 533–543 (1951).

This essay, essentially popular in its character, deals with philosophical problems concerning the connection between mathematics and physics. The author analyses the concepts of structure, number, then the problem of continuous or discrete character of space-time and finally that of probability.

L. Infeld (Warsaw).

ALGEBRA

***Bose, Raj Chandra. Mathematics of factorial designs.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 543–548. Amer. Math. Soc., Providence, R. I., 1952.

This is a lucid discussion of the theory of confounding in factorial designs and factorial designs with fractional replications. In the construction of such designs it is mostly necessary to assure that only higher order interactions are confounded and for this purpose orthogonal arrays are used. The author therefore discusses recent results in the theory of orthogonal arrays and gives two theorems by means of which several types of orthogonal arrays can be constructed.

H. B. Mann (Columbus, Ohio).

Shrikhande, S. S. On the non-existence of certain difference sets for incomplete group designs. Sankhyā **11**, 183–184 (1951).

Let $v = mn$. The author defines a difference set for an incomplete group design as a set of k residues $d_1, \cdots, d_k$ such that $d_i - d_j$ $(i \neq j)$ represents every residue mod mn which is divisible by n exactly λ_1 times and every other residue exactly λ_2 times. Let $\theta = k + (m-1)\lambda_1 - m\lambda_2$ and let ϕ be a prime dividing the squarefree part of θ and n a prime $\equiv 3(4)$. The author proves that no difference set as defined above can exist if $-n$ is quadratic non-residue mod ϕ.

H. B. Mann (Columbus, Ohio).

Shrikhande, S. S. On the non-existence of affine resolvable balanced incomplete block designs. Sankhyā 11, 185–186 (1951).

Affine resolvable designs have been defined by R. C. Bose [Sankhyā 6, 105–110 (1942); these Rev. 4, 237]. The author proves that the existence of a symmetrical design $v=b=n^2t+n+1$, $r=k=nt+1$, $\lambda=t$ and of an affine resolvable design $v=n^2(n-1)t+n^2$, $b=n(n^2t+n+1)$, $r=n^2t+n+1$, $k=n(n-1)t+n$, $\lambda=nt+1$ imply the existence of the symmetrical design $v=b=n(n^2t+n+1)+1$, $r=k=n^2t+n+1$, $\lambda=nt+1$. *H. B. Mann.*

***Littlewood, D. E. A University Algebra.** William Heinemann, Ltd., Melbourne, London, Toronto, 1950. viii+292 pp. 25 shillings.

The sixteen chapters of the book deal with the following topics: I. Linear equations and determinants; II. Matrices; III. Quadratic forms; IV. Groups; V. Symmetric functions; VI. Alternants and the general theory of determinants; VII. Further properties of matrices; VIII. Euclid's algorithm; IX. The laws of algebra; X. Polynomials; XI. Algebraic equations; XII. Galois theory of equations; XIII. Invariants; XIV. Algebras; XV. Group algebras—the symmetric group; XVI. The continuous groups.

It has been the author's endeavour to provide a textbook on algebra not only suitable for an Honours Degree course (with selected chapters for a Pass Degree course) but also for physicists interested in the alpplications of algebra to quantum theory and nuclear physics. In order to present in fewer than 300 pages such a vast material to so diverse classes of readers the author has been compelled to condense the arguments very severely. This has led in some cases to obscurity, in others to lack of rigour and incompleteness. To give two examples: the classical ideal theory for integers in an algebraic number field occupies three pages from the first definition of an ideal to the proof of unique factorisation. Galois theory is treated without mentioning the Jordan-Hölder theorem or the one-to-one correspondence between subgroups and between-fields. It appears very doubtful to the reviewer whether a student who is not already familiar with the theories can derive any benefit from trying to understand such a tour de force. In the elementary parts of the book there are some surprising omissions. (i) There is no account of the general theory of m linear equations in n unknowns; only fragments of the theory occur here and there. (ii) It is proved that a symmetric function can be expressed as a polynomial in the elementary symmetric functions, but the uniqueness of this expression is not proved. (Nevertheless the reader is referred to the method of undetermined coefficients!) (iii) In the theory of congruences residue classes are introduced and Fermat's theorem is proved for the modulus p. But Euler's function $\varphi(n)$ is not introduced (although on p. 200 primitive roots of congruences are employed without previous definition). (iv) In the theory of groups the concepts of conjugacy and of a normal subgroup occur, but the factor-group remains undefined. (Yet on p. 191 it makes its appearance, like a deus ex machina, in the formulation of Theorem III!)

The book shows many signs of careless writing. (i) In a number of places concepts are used which have not previously been defined. Apart from the examples quoted above we mention "order of a class" (p. 70), "degree" (p. 75), "weight and total degree" (p. 85), "primitive root of an equation" (p. 200). (ii) The author does not, as a rule, give axiomatic treatment to algebraic concepts. But in a few cases where axioms are introduced they are lax and incomplete. E.g. the axioms of "order" on p. 146—it should be called "Archimedean order"—do not ensure that the product of two positive elements is positive; and the axioms for an "algebra" on p. 224 do not ensure that the unit-element of the field is unit-operator—all rings according to the author's definition have unit-elements of multiplication, by the way. (iii) The concept of number remains vague throughout the book. Complex numbers occur explicitly for the first time on p. 43, but are not defined until p. 154; real numbers occur throughout the text, but on p. 146 a very brief definition of real numbers by means of Dedekind sections is given. Rules for the algebraic operations on reals are, however, not treated explicitly. It is, of course, perfectly justifiable from a didactic point of view to assume previous knowledge on the part of the reader, as long as the extent of such assumed knowledge is clearly stated. The lack of such a statement is precisely what the reviewer objects to.

A fair number of the definitions, theorems, and proofs in the book are faulty. We give here a few examples without claiming completeness. (i) According to the definition of linear dependence of vectors on p. 2 (repeated for an algebra over a field on p. 225) any set of vectors is linearly dependent, but the null vector is not dependent on any set of independent vectors. This confusion can, of course, be rectified by inserting, and omitting respectively, the non-zero condition on the coefficients. But the trouble is that some theorems (such as I and II in chapter I) depend on the wrong definition given by the author, while others (e.g. the equivalence between a vanishing determinant and a linear dependence among the row-vectors) require the correct definition. (ii) The author uses continuity arguments to obtain theorems for square matrices with some equal eigenvalues from those for matrices with distinct eigenvalues. This procedure can, of course, be justified in many cases (it is essentially H. Weyl's "principle of the irrelevance of algebraic inequalities"), but the author makes no attempt in this direction. It must be very bewildering for the undergraduate reader if the same principle applied to the two theorems: "Every matrix with distinct eigenvalues satisfies its characteristic equation" and "Every matrix with distinct eigenvalues can be transformed to diagonal form" leads in one case to a correct, in the other to an incorrect result. (iii) Theorem II on p. 156 states that the interchange of two roots of an irreducible equation is an automorphism of the extension field. This is manifestly wrong, as the author's own example of the fifth roots of unity demonstrates. (iv) Abel's theorem on the existence of a primitive element is proved on p. 158 without any restriction on the field F or the irreducible polynomial $f(x)$. This may be wrong for inseparable extensions. (v) The irreducibility of the cyclotomic equation for a prime number p is deduced on p. 200 from the fact that its Galois group is cyclic of order $p-1$. This is a petitio principii, because no separate proof is offered for this fact.

After so much criticism it is only fair to state that the last four chapters of the book are of a very different calibre. They contain an elegant account—though again in a highly condensed form far above undergraduate level—of those parts of algebra to which the author has contributed most in his research work.

K. A. Hirsch (London).

Popov, B. S. Sur une équation algébrique. Bull. Soc. Math. Phys. Macédoine 2, 3–15 (1951). (Macedonian. French summary)

The explicit solution of the equation

$$\sum_k \binom{n-k}{k} \frac{n}{n-k} p^k x^{2n-k} = -q$$

is given by observing that the left hand side is $z_1^n + z_2^n$ where $z^2 - xz - p = (z - z_1)(z - z_2)$. The result is applied to give the trigonometric solution of the reduced cubic and of the biquadratic to which a Tschirnhaus transformation has been applied. *R. Church* (Monterey, Calif.).

Ostrowski, Alexander. Ueber das Nichtverschwinden einer Klasse von Determinanten und die Lokalisierung der charakteristischen Wurzeln von Matrizen. Compositio Math. 9, 209–226 (1951).

Let the determinant $D = |(a_{\mu\nu})|$ of order n have real or complex elements with all $a_{\mu\mu} \neq 0$. Let $\alpha_{\mu\nu} = |a_{\mu\nu}|$; let $Z_\mu = -\alpha_{\mu\mu} + \sum_{\nu=1}^n \alpha_{\mu\nu}$; let $S_\mu = -\alpha_{\mu\mu} + \sum_{\nu=1}^n \alpha_{\nu\mu}$. For $0 \leqq \alpha \leqq 1$ let $M_\mu^{(\alpha)} = Z_\mu^\alpha S_\mu^{1-\alpha}$ and let $N_\mu^{(\alpha)} = \alpha Z_\mu + (1-\alpha) S_\mu \geqq M_\mu^{(\alpha)}$. Let λ_i denote the fundamental roots of the matrix of D.

Theorem I: If $\alpha_{\mu\mu} > M_\mu^{(\alpha)}$ for one α and all μ, then $D \neq 0$. Theorem II: If $\alpha_{\nu\nu}\alpha_{\mu\mu} > M_\nu^{(\alpha)} M_\mu^{(\alpha)}$ for one α and for all pairs ν, μ with $\nu \neq \mu$, then $D \neq 0$. Theorem III: For all α and i, λ_i lies in one of the n circles $K_\mu^{(\alpha)}$: $|\lambda - a_{\mu\mu}| \leqq M_\mu^{(\alpha)}$. Theorem IV: For all α and i,

$$|\lambda_i| \leqq \max_\mu (\alpha_{\mu\mu} + M_\mu^{(\alpha)}) \leqq \max_\mu (\alpha_{\mu\mu} + Z_\mu)^\alpha (\alpha_{\mu\mu} + S_\mu)^{1-\alpha} \leqq \max_\mu (\alpha_{\mu\mu} + N_\mu^{(\alpha)}).$$

Theorem V: For all α and i, λ_i satisfies one of the $\frac{1}{2}n(n-1)$ inequalities $|\lambda_i - a_{\nu\nu}| \cdot |\lambda_i - a_{\mu\mu}| \leqq M_\nu^{(\alpha)} M_\mu^{(\alpha)} (\mu \neq \nu)$. Additional theorems characterize the cases in which $D = 0$ when the inequalities of Theorems I and II are weakened to permit equality.

Theorem III may be strengthened as follows: If the set sum $K^{(\alpha)} = \sum_{\mu=1}^n K_\mu^{(\alpha)}$ consists of separated continua $\{K_j^{(\alpha)*}\}$, the number of λ_i lying in each $K_j^{(\alpha)*}$ is equal to the number of $a_{\mu\mu}$ lying in $K_j^{(\alpha)*}$. One can learn more by examining the intersections of the $K^{(\alpha)}$ for various α. A number of numerical examples ($n = 3$) with drawings illustrate the practical use of the extended Theorem III to separate the λ_i. Related use is made of Theorem V. There is a discussion of which values of α give the sharpest results in the various inequalities of Theorem IV. There are references to earlier versions of these theorems in the literature, limited to $\alpha = 0, \frac{1}{2}, 1$. *G. E. Forsythe* (Los Angeles, Calif.).

Sce, Michele. Su una generalizzazione delle matrici di Riemann. I. Ann. Scuola Norm. Super. Pisa (3) 5, 81–103 (1951).

The paper consists of an exposition in modern terminology of the elements of the theory of generalized Riemann matrices as given by H. Weyl [Ann. of Math. (2) 35, 714–729 (1934)] and the reviewer [ibid. 36, 886–964 (1935)].

A. A. Albert (Chicago, Ill.).

Gaeta, Federigo. Sur la distribution des degrés des formes appartenant à la matrice de l'idéal homogène attaché à un groupe de N points génériques du plan. C. R. Acad. Sci. Paris 233, 912–913 (1951).

The homogeneous polynomial ideal defined by N generic points in a plane has a base consisting of the determinants of order $\rho+1$ contained in a matrix of $\rho+1$ rows and $\rho+2$ columns (ρ being a non-negative integer) where elements are homogeneous polynomials. This matrix is supposed normalised so that the degrees of the polynomials decrease (in the broad sense) down the first column and increase (in the broad sense) along the first row. Considerations of homogeneity determine the degree of an element of the matrix in terms of elements in the first row and first columns. The author shows that the polynomials are all of the first or second degree, and describes a simple algorithm by which, N being given, ρ and the degrees of the polynomial may be determined. *J. A. Todd* (Cambridge, England).

Abstract Algebra

Haimo, Franklin. Some limits of Boolean algebras. Proc. Amer. Math. Soc. 2, 566–576 (1951).

In this paper the author considers the notions of inverse and direct limits of indexed sets of Boolean algebras modelled according to those already in use for topological spaces and groups. The discussion centers around the behaviour of ideals and some decompositions of the algebras. The author also investigates what happens to the Stone representation spaces of the algebras when either kind of limit is taken. *L. Nachbin* (Rio de Janeiro).

Ribeiro, Hugo. A remark on Boolean algebras with operators. Amer. J. Math. 74, 163–167 (1952).

Jónsson and Tarski [same J. 73, 891–939 (1951); these Rev. 13, 426] considered three classes of functions on Boolean algebras, $A \subset \phi \subset M$, where A is the class of additive (resp. union) functions, M of monotone functions, and ϕ the class generated by additive functions under composition

$$f((g_1(x_1, \cdots, x_n), \cdots, g_m(x_1, \cdots, x_n)) = f[g](x).$$

A natural extension of f on the algebra $\mathfrak{A}$ to f^+ on its perfect extension $\mathfrak{A}^+$ was defined and a central result was that $f[g]^+(x) = f^+[g^+](x)$ for $f, g_1, \cdots, g_n$ in ϕ. Ribeiro establishes this under the weaker hypothesis f in ϕ, $g_1, \cdots, g_n$ in M, and shows incidentally that, for x closed, $f, g_1, \cdots, g_n$ in M is sufficient. *R. C. Lyndon* (Princeton, N. J.).

Jónsson, Bjarni, and Tarski, Alfred. Boolean algebras with operators. II. Amer. J. Math. 74, 127–162 (1952).

The concepts of Part I [same J. 73, 891–939 (1951); these Rev. 13, 426] are applied to relation algebras. A 'relation algebra' (r.a.) is a boolean algebra (universal element V) with a binary operation $x \cdot y$ (we depart from Schröder's notation, used in the paper), associative and with neutral element I, and a unary operation $x\smile$ subject to

$$x \cdot y \cap z = 0 \rightleftarrows x\smile \cdot z \cap y = 0 \rightleftarrows z \cdot y\smile \cap x = 0.$$

A 'proper' r.a. is a boolean algebra of binary relations over some domain, in which $x \cdot y$ and $x\smile$ mean relative product and converse. Products, homomorphic images, and subalgebras are studied, and it is proved that every r.a. is a subdirect product of simple r.a.'s. By the main result of part I, every r.a. can be embedded in a complete atomistic r.a. For such, a weak representation is defined: $uR_xv \rightleftarrows u \subset x \cdot v$ (u, v atoms), in which intersection is not preserved. [Isomorphic representation is not always possible; see Lyndon, Ann. of Math. (2) 51, 707–729 (1950); these Rev. 12, 237.] A proper r.a. is always isomorphic to one in which I is a true identity relation; and V a true universal relation if and only if the algebra is simple.

Special classes of r.a.'s are now considered. Every complete atomistic algebra in which every atom satisfies $x\smile\cdot V\cdot x\subset I$ is isomorphic to the full proper algebra over some domain [cf. McKinsey, J. Symbolic Logic 5, 85–97 (1940); these Rev. 2, 66]. Every r.a. is decomposed into $\mathfrak{A}_0\oplus\mathfrak{A}_I\oplus\mathfrak{A}_{I'}$, where $\mathfrak{A}_\theta$ satisfies $I'\cdot I'=\theta$ (I'='diversity'; $\theta=0, I, I'$) and $\mathfrak{A}_0$ ($\mathfrak{A}_I$) is isomorphic to a subdirect sum of proper algebras on domains of one (two) elements.

'Functional' algebras (reviewer's terminology) are atomistic, with every atom satisfying $x\smile\cdot x\subset I$. A (generalized Brandt) groupoid $\mathfrak{U}$ is a Brandt groupoid with $(x)(z)(\exists y).x\cdot y\cdot z\varepsilon\mathfrak{U}$ dropped. The 'complex algebra' $\mathfrak{U}^*$ (cf. part I) is a functional r.a., and every functional algebra $\mathfrak{A}$ is representable as a subalgebra of some $\mathfrak{U}^*$. $\mathfrak{A}$ simple means $\mathfrak{U}$ Brandt; $\mathfrak{A}$ complete, $\mathfrak{A}\simeq\mathfrak{U}^*$. If the y, above, is unique, $\mathfrak{A}$ is isomorphic to the algebra of all subrelations of some equivalence V.

'Integral' algebras are characterized by $x\cdot y=0$ implies $x=0$ or $y=0$, and are necessarily simple. A functional algebra is integral just in case the groupoid $\mathfrak{U}$ is a group. Whether every integral algebra, even if assumed proper, is isomorphic to the complex algebra of some group, remains an open question. *R. C. Lyndon* (Princeton, N. J.).

Kořínek, Vladimír. Lattices in which the theorem of Jordan-Hölder is generally true. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 50 (1949), 307–324 (1951).

English version of an earlier paper in Czech [Rozpravy II. Třídy České Akad. 59, no. 23 (1949); these Rev. 12, 667].

Hostinsky, L. Aileen. Direct decompositions in lattices. Amer. J. Math. 73, 741–755 (1951).

Die Baersche Verallgemeinerung [Trans. Amer. Math. Soc. 62, 62–98 (1947); diese Rev. 9, 134] des Wedderburnschen Verfeinerungssatzes für direkte Zerlegungen wird rein verbandstheoretisch behandelt. Verf. benutzt darin ihre Untersuchungen über "Endomorphisms of lattices" [Duke Math. J. 18, 331–342 (1951); diese Rev. 12, 795]. Es wird ein vollständiger modularer Verband mit einer schwachen Volldistributivitätseigenschaft zugrundegelegt und unter Voraussetung der etwas modifizierten "splitting hypothesis" von Baer [loc. cit.] der Verfeinerungssatz bewiesen. Die Beweise sind allerdings z. T. erheblich geändert. Die splitting hypothesis ist schwächer als die Oresche Bedingung [Duke Math. J. 2, 581–596 (1936)] für den Verfeinerungssatz. *P. Lorenzen* (Bonn).

Iseki, Kiyoshi. A characterization of distributive lattices. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 388–389 (1951).

Let L be a lattice with 0. For L to be distributive, it is necessary and sufficient that for every pair of distinct elements of L there exist a prime ideal containing one of the pair, the other element of the pair belonging to the complementary ideal. Likewise, it is necessary and sufficient that, given two distinct elements of L, there exist a two-valued isotone valuation on L having distinct values for the given elements. [The necessity of these conditions was shown by M. H. Stone, Časopis Pěst. Mat. Fys. 67, 1–25 (1937).]
P. M. Whitman (Silver Spring, Md.).

Arnold, B. H. Distributive lattices with a third operation defined. Pacific J. Math. 1, 33–41 (1951).

Convenons d'appeler $*$-treillis un treillis distributif muni, outre les deux opérations habituelles $\vee$ et $\wedge$, d'une troisième opération $*$, idempotente, commutative, associative, distributive par rapport à $\vee$ et à $\wedge$. (Un $*$-treillis entre donc dans la classe des m-lattices de G. Birkhoff.) Le produit direct de deux $*$-treillis est un $*$-treillis. L'auteur démontre que tout $*$-treillis est isomorphe à un sous-$*$-treillis S du produit direct de deux $*$-treillis bien déterminés et si $(a, b)\varepsilon S$ et $(c, d)\varepsilon S$, $(a, b)*(c, d)=(a\wedge c, b\vee d)$. Un intéressant rapprochement entre la structure de $*$-treillis et la structure d' "ordre double" utilisée par Stöhr [J. Reine Angew. Math. 184, 138–157 (1949); ces Rev. 5, 226] n'est que suggéré par la bibliographie. *J. Riguet* (Paris).

Fodor, G. On a problem concerning the theory of binary relations. Nieuw Arch. Wiskunde (2) 23, 247–248 (1951).

Soit I l'intervalle fermé $[0, 1]$ et soit R une relation binaire entre éléments de I c'est-à-dire un sous-ensemble R de l'ensemble $I\times I$ de tous les couples d'éléments de I. On désigne par $R(x)$ l'ensemble des $y\varepsilon I$ tels que $(x, y)\varepsilon R$. L'auteur montre qu'en général, si R n'est astreint qu'à la seule condition que pour tout x, le complémentaire de $R(x)$ contienne un intervalle dont x est le milieu et dont la longueur est non nulle, il ne peut exister de sous-ensemble A de I, de mesure positive, tel que $(*)$ $R\cap(A\times A)=\phi$. Ce résultat et sa démonstration seraient à rapprocher de ceux de D. Lázár [Compositio Math. 3, 304 (1936)] relatifs à l'existence d'un sous ensemble A ayant la puissance du continu et satisfaisant à $(*)$. *J. Riguet* (Paris).

Petropavlovskaya, R. V. On the decomposition into a direct sum of the structure of subsystems of an associative system. Doklady Akad. Nauk SSSR (N.S.) 81, 999–1002 (1951). (Russian)

Let A be an associative system (that is, closed under an associative multiplication) and denote by $\{x\}$ the subsystem generated by x. If A contains an idempotent e and τ is a set of subsystems of A, then write $A=\prod_e\tau$ if the following hold: (1) $B\varepsilon\tau$ implies $B\neq\{e\}$; the intersection of B with any subsystem, generated by elements of members of τ other than B, is $\{e\}$; (2) if $b_1\varepsilon B_1\varepsilon\tau$ then $b_1e=eb_1$; (3) if also $b_2\varepsilon B_2\varepsilon\tau$ then $b_1b_2e=b_2b_1e$; (4) if $x\varepsilon A$ but x non-ε B for each $B\varepsilon\tau$ then $x=b_1\cdots b_ne$ for $b_i\varepsilon B_i\varepsilon\tau$. Assume that the set $P(A)$ of all non-empty subsystems of A is a complete lattice. Theorem: $P(A)$ is the direct sum of a set π of its sublattices if and only if there is a set τ of non-empty subsystems of A satisfying (1) π is the set of lattices $P(B)$, $B\varepsilon\tau$; (2) there exists $e\varepsilon A$ with $ee=e$ and $A=\prod_e\tau$; (3) if $b\varepsilon B$, $B\varepsilon\tau$, and x is an element of the subsystem generated by all the subsystems, other than B, belonging to τ then $bx\varepsilon B$ implies $bx\varepsilon\{b\}$, and $xb\varepsilon B$ implies $xb\varepsilon\{b\}$; (4) if $b_1\varepsilon B_1\varepsilon\tau$ and $b_2\varepsilon B_2\varepsilon\tau$ then $\{b_1e\}$ and $\{b_2e\}$ have finite relatively prime numbers of elements. Theorem: If $P(A)$ is the direct sum of sublattices, then there is a unique most refined such decomposition. Applied to groups, these results specialize to known theorems. All proofs are omitted. *P. M. Whitman.*

Szélpál, I. Über Ringerweiterungen. Acta Sci. Math. Szeged 14, 113–114 (1951).

Snapper, E. Completely primary rings. IV. Chain conditions. Ann. of Math. (2) 55, 46–64 (1952).

The same notation is used as in the three previous papers of this series [Ann. of Math. (2) 52, 666–693 (1950); 53, 125–142 (1951); 53, 207–234 (1951); these Rev. 12, 314, 387, 584]. In particular, A and B represent arbitrary commutative rings with unit element, while R and S are completely primary rings. If $A\subset B$ and for every ideal $\mathfrak{a}$ in A the con-

traction of the extension of $\mathfrak{a}$ is equal to $\mathfrak{a}$, we say that the (e, c)-condition is satisfied in the ring extension $A \subset B$. The ring A is called a primitive ring if the (e, c)-condition is satisfied for every ring extension of A. The ring A is a self-dual ring if every ideal of A is an annihilator. Every self-dual ring is primitive, and the converse is also true for completely primary rings with chain conditions. A principal ring extension which satisfies the (e, c)-condition is called a canonical extension. Let $S \subset S'$, where S has a composition series of length λ. If the (e, c)-condition is satisfied in this extension, S' is a canonical extension of S if and only if S' has a composition series of length λ. If $R \subset S$ is a canonical extension and the ideals of R satisfy the chain conditions, each Loewy series of R becomes the corresponding Loewy series of S under ideal extension and each Loewy series of S becomes the corresponding Loewy series of R under ideal contraction. The rings R and S have the same Loewy invariants. Now if R and S are arbitrary with $R \subset S$, R is called a coefficient ring of S if R is a canonical extension of the prime ring of S and $\bar{R} = \bar{S}$. If $N(S)$ has a finite ideal basis, S has a coefficient ring R and, considered as an R-module, S satisfies both chain conditions. The author studies such modules and, in particular, applies the results to the case of simple algebraic extensions of principal ideal rings. The final part of the paper consists of a detailed study of the situation which holds when S is a finite extension of R, and both R and S are principal ideal rings.

N. H. McCoy (Northampton, Mass.).

Wyler, Oswald. Ueber einen Rangbegriff in der Theorie der Ringe, speziell der regulären Ringe. Compositio Math. **9**, 193–208 (1951).

A definition of rank is introduced for elements of an arbitrary associative ring, in particular for regular rings, generalizing the rank definition of square matrices. It is observed that certain statements concerning ranks of matrices can be formulated for arbitrary rings; e.g. two $n \times n$ matrices which generate the same left (right) ideal in the ring of all $n \times n$ matrices have the same rank; conversely, if two matrices a, b are of the same rank then an $n \times n$ matrix c exists which generates the same right ideal as a and the same left ideal as b. The latter statement is used as a definition for equal rank for two elements in an arbitrary ring; a number of equivalent definitions are given. The rank itself is defined as a certain set of elements. Ranks cannot always be added, but if addition is possible it is an associative and commutative operation. The author uses the results of R. Baer [Amer. J. Math. **71**, 706–742 (1949); these Rev. **11**, 78] concerning such systems. He constructs an extension of the set of ranks to a set closed under an associative addition. If the ring is regular, this addition is further commutative. The results are used to obtain proofs for theorems concerning semisimple rings [cf. Kaplansky, Trans. Amer. Math. Soc. **68**, 62–75 (1950); these Rev. **11**, 317] and for a construction of the matrix ring isomorphic to a simple ring satisfying the minimum condition for left ideals.

O. Taussky-Todd (Washington, D. C.).

Brandt, Heinrich. Allgemeine Modultheorie. Hallische Monographien no. **22**, pp. 3–10. Max Niemeyer Verlag, Halle (Saale), 1951. 4.00 D.M.

This paper, written in 1927, gives the basic ideas of the non-commutative arithmetic theory. If $\mathfrak{S}$ is a ring whose regular elements form a multiplicative group, and if $\mathfrak{z}$ is a subring of the center of $\mathfrak{S}$ containing the identity element, then this paper is a study of the $\mathfrak{z}$ modules contained in $\mathfrak{S}$. The sum, product, intersection, and quotient of $\mathfrak{z}$ modules are defined, and their elementary properties are given.

R. E. Johnson (Northampton, Mass.).

✓ ***Krull, Wolfgang. Jacobsonsches Radikal und Hilbertscher Nullstellensatz.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 56–64. Amer. Math. Soc., Providence, R. I., 1952.

Let $\mathfrak{R}$ be a commutative ring with unity. The Jacobson radical $\mathfrak{r}_j(\mathfrak{a})$ of an ideal $\mathfrak{a}$ of $\mathfrak{R}$ is the intersection of all maximal ideals of $\mathfrak{R}$ which contain $\mathfrak{a}$; the radical $\mathfrak{r}(\mathfrak{a})$ of $\mathfrak{a}$ is the intersection of all minimal prime overideals of $\mathfrak{a}$; obviously $\mathfrak{r}_j(\mathfrak{a})$ contains $\mathfrak{r}(\mathfrak{a})$. The author defines $\mathfrak{R}$ to be a Jacobson ring if $\mathfrak{r}_j(\mathfrak{a}) = \mathfrak{r}(\mathfrak{a})$ for every $\mathfrak{a}$. Theorem (generalizing the Hilbert theorem on zeros): If $\mathfrak{R}$ is a Jacobson ring then so is every finite ring extension $\mathfrak{S} = \mathfrak{R}[\alpha_1, \cdots, \alpha_n]$, and if K is a homomorphism of $\mathfrak{S}$ onto a field then $K(\mathfrak{R})$ is a field and $K(\mathfrak{S})$ is algebraic over $K(\mathfrak{R})$. Theorem: Let $\mathfrak{R}$ be a Noetherian Jacobson ring, $\mathfrak{p}$ a prime ideal of the polynomial ring $\mathfrak{R}[x_1, \cdots, x_n]$, d the dimension of $\mathfrak{p}$, $d^{(r)}$ the dimension of $\mathfrak{p} \cap \mathfrak{R}$; if $d^{(r)} = \infty$ then $d = \infty$, if $d^{(r)} < \infty$ then $d^{(r)} \leqq d \leqq d^{(r)} + n$. Certain conjectures and the relations among them are discussed.

E. R. Kolchin.

✓ ***Nakayama, Tadasi. On two topics in the structural theory of rings (Galois theory of rings and Frobenius algebras).** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 49–54. Amer. Math. Soc., Providence, R. I., 1952.

An exposition of the results of several recent papers of the author [especially Canadian J. Math. **3**, 208–218 (1951); these Rev. **13**, 8; (with G. Azumaya) Ann. of Math. (2) **48**, 949–965 (1947); these Rev. **9**, 563; ibid. **40**, 611–633 (1939); **42**, 1–21 (1941); Jap. J. Math. **18**, 49–65 (1942); these Rev. **1**, 3; **2**, 344, **7**, 361] together with a valuable general report on the present state of the two subjects mentioned.

G. Whaples (Bloomington, Ind.).

Bruck, R. H., and Kleinfeld, Erwin. The structure of alternative division rings. Proc. Amer. Math. Soc. **2**, 878–890 (1951).

This is the authors' proof of the important and surprising result that an alternative division ring of characteristic not 2 is either i) associative or ii) a Cayley-Dickson algebra over its center. An independent proof was also found by Skornyakov [Ukrain. Mat. Žurnal **2**, 70–85 (1950); these Rev. **12**, 668], which, using different methods, gave the result for characteristic not 2 or 3.

The commutator (x, y) and associator (x, y, z) are defined for a general distributive ring R by the rules $(x, y) = xy - yx$, $(x, y, z) = (xy)z - x(yz)$. These are functions linear in each argument. R is an alternative ring if these functions are skew-symmetric, i.e. vanish whenever two of the arguments are equal. The authors proceed to find identities for R by finding further skew-symmetric multilinear functions. The first of these is $f(w, x, y, z) = (wx, y, z) - (x, y, z)w - x(w, y, z)$. Another function $g(u, v, w, x, y)$ is defined which is skew-symmetric for the triple u, v, w and the pair x, y. Having established the skew-symmetry for f and g, they easily obtain all the identities they need, including the Moufang identities.

The nucleus N of R is the set of n with $(n, R, R) = 0$. The center C is the set of c with $(c, R) = 0 = (c, R, R)$. The first main step in the proof is the demonstration that if R is an

alternative ring without divisors of zero, then $N=R$ or $N=C$. The next major step consists in showing that if $(x, y, z)\neq 0$ (and hence $N\neq R$), then $px^2-qx+r=0$ with p, q, r in C. Here application of a result of A. A. Albert shows that R is a Cayley-Dickson algebra of order 8 over C.

As an application, a question raised by the reviewer is answered. Ruth Moufang has shown that a projective plane which satisfies the theorem of the Complete Quadrilateral may be coordinatized by an alternative division ring and conversely. The reviewer gave a condition that different rings coordinatizing the same plane be isomorphic, but raised the question as to whether this condition was satisfied for alternative, non-associative division rings. R. D. Schafer had previously shown that this condition held for Cayley-Dickson algebras. Since the present paper shows these algebras to be the only non-associative division rings, the reviewer's question is answered affirmatively.

Marshall Hall (Washington, D. C.).

Kleinfeld, Erwin. Alternative division rings of characteristic 2. Proc. Nat. Acad. Sci. U. S. A. **37**, 818–820 (1951).

The present paper completes the investigations of that above. It uses the results above and an idea suggested by Skornyakov's paper. A Cayley-Dickson algebra of characteristic 2 contains elements a, b, c which permute pairwise although $(a, b, c)\neq 0$. For an arbitrary non-associative alternative division ring of characteristic 2 he finds such a triple a, b, c. Then he shows that if

$$(p, a, b)=(p, b, c)=(p, a, c)=0,$$

then p is in the center C of R. With this lemma he proceeds to show that $1, a, b, c, ab, ac, bc, a(bc)$ form a basis for R over C, thus representing R explicitly as a Cayley-Dickson algebra.

Marshall Hall (Washington, D. C.).

Schafer, R. D. Representations of alternative algebras. Trans. Amer. Math. Soc. **72**, 1–17 (1952).

Let $\mathfrak{B}$ be a vector space over $\mathfrak{F}$ and $\mathfrak{A}$ be an alternative algebra over $\mathfrak{F}$. Then a representation (S, T) of $\mathfrak{A}$ is a pair of mappings $x\rightarrow S_x$, $x\rightarrow T_x$ of $\mathfrak{A}$ into the set of all linear transformations on $\mathfrak{A}$ satisfying

$$T_xS_z-S_zT_x=S_{xz}-S_xS_z=T_{zx}-T_zT_x=S_xT_z-T_zS_x$$

for all x and z in $\mathfrak{A}$. If $\mathfrak{M}$ is any subspace of $\mathfrak{A}$ we denote by $S(\mathfrak{M})$, $T(\mathfrak{M})$, $U(\mathfrak{M})$ respectively the sets of all S_x, T_x and S_x+T_y for x and y in $\mathfrak{M}$. Let the enveloping algebras of these sets be designated by $S(\mathfrak{M})^*$, $T(\mathfrak{M})^*$, $U(\mathfrak{M})^*$. The author shows that if $\mathfrak{A}$ has characteristic zero and $\mathfrak{N}$ is its radical, then $S(\mathfrak{N})$ is contained in the radical of $S(\mathfrak{N})^*$ with corresponding results for $T(\mathfrak{N})$ and $U(\mathfrak{N})$. Using these facts, he shows that if $\mathfrak{A}$ is semisimple any representation (S, T) of $\mathfrak{A}$ is completely reducible. The first Whitehead lemma for associative algebras is generalized to alternative algebras and a stronger form is proved which is equivalent to the statement that any derivation of $\mathfrak{A}$ into $\mathfrak{B}$ can be extended to an inner derivation of $\mathfrak{B}$. As a consequence he proves the Malcev theorem for alternative algebras on the strict conjugacy of $\mathfrak{S}$ and $\mathfrak{S}_1$ in any two Wedderburn decompositions $\mathfrak{A}=\mathfrak{S}+\mathfrak{N}=\mathfrak{S}_1+\mathfrak{N}$ of $\mathfrak{A}$. The final result of the paper is the theorem which states that $\mathfrak{A}$ is semisimple if and only if its derivation algebra is semisimple.

A. A. Albert.

Dynkin, E. B. On semisimple subalgebras of semisimple Lie algebras. Doklady Akad. Nauk SSSR (N.S.) **81**, 987–990 (1951). (Russian)

The author determines the simple subalgebras (within conjugacy) of all the exceptional simple Lie algebras over the complex field, thereby completing the work of Malcev [Izvestiya Akad. Nauk SSSR. Ser. Mat. **8**, 143–174 (1944); these Rev. **6**, 146] who had made this determination for the classical simple Lie algebras and also for G_2 and partially for F_4. He uses in this determination some general results about semi-simple subalgebras of semi-simple Lie algebras which he derives.

I. E. Segal (Chicago, Ill.).

✓***Albert, A. A. Power-associative algebras.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 25–32. Amer. Math. Soc., Providence, R. I., 1952.

Exposé d'ensemble de la théorie des algèbres non associatives où toute sous-algèbre engendrée par un élément quelconque est associative; cette théorie a été surtout développée par l'auteur, qui s'est attaché principalement à déterminer les algèbres de cette catégorie qui sont simples: l'auteur entend par là les algèbres non nilpotentes et n'ayant que des idéaux bilatères triviaux (on ignore si la première condition est conséquence de la seconde pour les algèbres de dimension >1). Parmi les résultats les plus importants passés en revue dans cet exposé, mentionnons la théorie des algèbres simples commutatives [développée dans Trans. Amer. Math. Soc. **69**, 503–527 (1950); ces Rev. **12**, 475], la théorie des algèbres de Jordan (où la détermination des algèbres simples est due à l'auteur, à G. Kalisch et N. Jacobson), et enfin celle des algèbres admettant une "trace" due à l'auteur [Proc. Nat. Acad. Sci. U. S. A. **35**, 317–322 (1949); ces Rev. **11**, 6]. L'auteur utilise à plusieurs reprises la méthode qui consiste à associer à une algèbre A l'algèbre commutative A^+ obtenue en prenant comme nouvelle multiplication la loi $x\cdot y=\frac{1}{2}(xy+yx)$; il indique en terminant qu'il a déterminés les algèbres simples telles que A^+ soit une algèbre de Jordan centrale simple et spéciale (c'est-à-dire plongée dans une algèbre de Jordan G^+ provenant d'une algèbre G associative).

J. Dieudonné (Nancy).

Glaeser, Georges. Dérivation des algèbres commutatives. C. R. Acad. Sci. Paris **233**, 1550–1552 (1951).

By a derivation of a commutative algebra A over a field K the author means a pair u, D of linear functions on A (values in K) such that $D(fg)=u(f)D(g)+u(g)D(f)$ for all f, g in A. If $D\neq 0$ there exists a λ_0 of K such that $u(fg)=u(f)u(g)+\lambda_0 D(f)D(g)$. When λ_0 is a square μ^2 (μ in K, $\mu\neq 0$) then $u+\mu D$, $u-\mu D$ are characters of the algebra A; when this is not the case, one constructs a quadratic extension of K, either an overfield of K or an algebra of dual numbers ($\mu^2=0$), and $u+\mu D$, $u-\mu D$ are conjugate generalized characters of A. The hyperplane $\mathfrak{D}$ defined by $D(f)=0$ is a subalgebra; the locus of $D(f^2)=0$ is a union $\mathfrak{D}\cup\mathfrak{U}$ of $\mathfrak{D}$ with a hyperplane $\mathfrak{U}$, and $\mathfrak{D}\cap\mathfrak{U}$ is an ideal. As every subalgebra of codimension 1 can be used to define a derivation, it follows that every subalgebra of codimension 1 which is not an ideal contains an ideal of codimension 2.

E. R. Kolchin (New York, N. Y.).

Kawai, Ryôichirô. Notes on the F. K. Schmidt's "Quasidifferente" in function-fields. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 145–147 (1951).

In his proof of the theorem of Riemann-Roch for fields of algebraic functions over arbitrary fields of constants, F. K. Schmidt [Math. Z. **41**, 415–438 (1936)] has introduced the notion of a quasi-different, which is an ideal $\vartheta_x(L/K)$, associated with a non-constant element x of the function field L over the constant field K, and which can be taken to be the Dedekind different of $L/K(x)$ if L is separable over

$K(x)$. In general, for a given x, the quasi-different is determined only up to a principal ideal. As far as one can merely guess from the present sketchy and obscurely written note, the author shows by an elementary consideration involving parastrophic matrices that, if $L\supset M\supset K(x)$, there is a relation $\vartheta_s(M/K)\delta_s(L/M)=\vartheta_s(L/K)$, where $\delta_s(L/M)$ is an ideal of L which is determined up to a principal ideal. From F. K. Schmidt's analogue of Riemann's formula for the genus, it is then seen that, if g_L and g_M denote the genus of L and M, respectively, $2g_L-2=\deg(\delta_s(L/M))+[L:M](2g_M-2)$.

G. Hochschild (New Haven, Conn.).

Theory of Groups

Piccard, Sophie. Structure des groupes imprimitifs. Suites associées, classes de substitutions, sous-groupes distingués, nombre minimum d'éléments générateurs. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 297–307 (1951).

This paper contains the proofs of theorems already stated by the author [cf. C. R. Acad. Sci. Paris **231**, 14–16 (1950); these Rev. **12**, 9] concerning the properties of a sequence $G_1, G_2, \cdots, G_m$ of permutation groups associated with an imprimitive group G_1. The group G_m is by definition primitive. The integer m is not uniquely defined and may, for a given G_1, take any value h where $2\leqq h\leqq m$. Theorem I generalizes the familiar distinction between even and odd permutations so that it involves the associated permutations of the groups G_i. Theorem II proves the existence of a group G_1 for which m takes any preassigned value and G_1 is of order $2^{2^m}-1$. In conclusion, the author ties in these ideas with her previous work on the number of independent generators of a group and shows that this number for G_1 is equal to or greater than m.

G. de B. Robinson.

Rédei, L. Die Einfachheit der alternierenden Gruppe. Monatsh. Math. **55**, 328–329 (1951).

If A is the alternating group of $n>4$ digits and N is a normal subgroup of A, then $N\supseteq(A, N)$. From this follows by simple combinatorial methods that N contains every element of the type $(a\,b)(c\,d)$. These elements generate A; hence $N=A$.

F. W. Levi (Bombay).

Ospanov, A. On a problem of the theory of finite groups. Izvestiya Akad. Nauk Kazah. SSR. **60**, Ser. Mat. Meh. **3**, 91–100 (1949). (Russian. Kazak summary)

In connection with the Burnside problem on the existence of a simple group of odd composite order, the author finds there are no such groups when the order has one of the three forms: p^5qrs, p^4q^2rs, p^3q^3rs, where p, q, r, s are distinct odd primes.

R. A. Good (College Park, Md.).

Coxeter, H. S. M. The product of the generators of a finite group generated by reflections. Duke Math. J. **18**, 765–782 (1951).

A remarkable relationship is found between the degrees of the basic invariant forms of a finite group G, generated by n reflections $R_1, R_2, \cdots, R_n$, and the characteristic roots of the product R of these reflections. Namely, if R is of period h, if $\omega=e^{2\pi i/h}$, and if the n roots of R are ω^{m_j}, $(0\leqq m_j<h)$, then the integers m_j+1 are equal to the degrees of the basic invariant forms. Just why this should be so is not fully understood, but the fact is proved by computing these roots for every one of the finite groups generated by reflections and comparing the values of m_j+1 with the computed degrees of the basic invariants. A proof supplied by G. Racah is used to show that $\prod(m_j+1)$ is the order of the group, and $\sum m_j=nh/2$ is the number of reflecting hyperplanes. The quantities $1-\cos(m_j\pi/h)$, related to the roots ω^{m_j}, are shown to be characteristic roots of the symmetric matrix (a_{ik}), whose entry a_{ik} is the cosine of the angle between the directed normals to the ith and kth of the chosen n reflecting hyperplanes. When the only angles involved are 90° and 120°, the determinantal equation is expressible in terms of Chebyshev polynomials. The geometry of the groups is studied in some detail for $n=2, 3$. Then the degrees of the basic invariants are computed. Finally, it is observed that when the group is crystallographic, the Betti numbers of the group manifold of the corresponding continuous group are coefficients in the Poincare polynomial $\prod(1+t^{2m_j+1})$ in which the same integers m_j play an essential role.

J. S. Frame (East Lansing, Mich.).

Hughes, N. J. S. The use of bilinear mappings in the classification of groups of class 2. Proc. Amer. Math. Soc. **2**, 742–747 (1951).

In his paper "The classification of prime-power groups" [J. Reine Angew. Math. **182**, 130–141 (1940); these Rev. **2**, 211] P. Hall has introduced the equivalence relation of "isoclinism" between groups and the division into families of isoclinic groups. In the present paper the author proves that there exists a one-to-one correspondence between the families of groups of class 1 (i.e. the single family of Abelian groups) and class 2 on the one hand, and the families of "regular bilinear mappings" on the other. A regular bilinear mapping of a group H into a group K (H and K turn out to be Abelian) is defined as a function $f(x, y)$ of two ordered arguments in H with values uniquely defined in K which satisfies the following conditions: (i) $f(xx', y)=f(x, y)f(x', y)$ and similarly for the second argument, (ii) $f(x, x)=e$, the group identity, (iii) if $f(x, y)=e$ for all y, then $x=e$, and (iv) K is generated by the values of f. If now G is a group of class 2, Z its centre and Q its derived group, $Q\subseteq Z$, then the mapping of the centre quotient group G/Z into the derived group Q defined by $f(Zx, Zy)=x^{-1}y^{-1}xy$ is a regular bilinear mapping and is denoted by $M(G)$. Two mappings f and f' are said to belong to the same family, if the corresponding groups H and H', K and K' are isomorphic in such a manner that if $H'=\varphi H$ and $K'=\psi K$, then $f'(\varphi x, \varphi y)=\psi f(x, y)$. Isoclinism between two groups G and G' is then exactly the same as the fact that the mappings $M(G)$ and $M(G')$ belong to the same family. The proof of the main theorem mentioned above establishes the existence, for any regular bilinear mapping f, of a group G such that f and $M(G)$ belong to the same family and that, moreover, $Q=Z$ (so that G is a stem-group in the sense of P. Hall). In the second part a direct product of families is defined and it is shown that the family of any finite group of class 2 can be split uniquely into the product of indecomposable mappings.

K. A. Hirsch (London).

Tartakovskiĭ, V. A. The sieve method in group theory. Application of the sieve method to the solution of the word problem for certain types of groups. Solution of the word problem for groups with a k-reduced basis for $k>6$. Amer. Math. Soc. Translation no. **60**, 110 pp. (1952).

These papers are translated from Mat. Sbornik N.S. **25**(67), 3–50, 251–274 (1949); Izvestiya Akad. Nauk SSSR. Ser. Math. **13**, 483–494 (1949); these Rev. **11**, 493.

Parker, E. T. On a question raised by Garrett Birkhoff. Proc. Amer. Math. Soc. 2, 901 (1951).

Garrett Birkhoff [Lattice theory, rev. ed., Amer. Math. Soc. Colloq. Publ. vol. 25, New York, 1948, p. 98; these Rev. 10, 673] has shown that every group whose order is the product of $k \leqq 4$ primes has subgroup lattice of length k. He proposed the problem (number 39) to determine the largest integer k such that every group whose order is the product of k primes has subgroup lattice of length k. The author here shows that the largest such integer is 4. The proof is based on the demonstration that the simple group of order $2^2 \cdot 3 \cdot 7 \cdot 13$ [Burnside, Theory of groups of finite order, Cambridge Univ. Press, 1897, pp. 338, 367] violates the required condition. *D. C. Murdoch.*

Sato, Shoji. Note on lattice-isomorphisms between Abelian groups and non-Abelian groups. Osaka Math. J. 3, 215–220 (1951).

The main result of this paper consists in the proof of the following theorem. If the lattice of subgroups of the group G is modular, if the subgroup $F(G)$ of elements of finite order in G is abelian, and if $G/F(G)$ is an abelian group of rank 1, then G is lattice-isomorphic to an abelian group.

R. Baer (Urbana, Ill.).

Pic, Gheorghe. On the structure of quasi-Hamiltonian groups. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 973–979 (1949). (Romanian. Russian and French summaries)

A subgroup of a group is called quasi-normal if it is permutable with any other subgroup [Ore, Duke Math. J. 3, 149–174 (1937)]. The author calls a group quasi-hamiltonian, if all its subgroups are quasi-normal. The structure of such groups is elucidated by the following theorems: 1. Every finite quasi-hamiltonian group is nilpotent, i.e. the direct product of its Sylow-subgroups. 2. The p-Sylow-subgroup of a finite quasi-hamiltonian group is either Abelian, or the direct product of an Abelian group and a group generated by two elements, say s and t of orders p^σ and p^τ. 3. Between s and t there holds a commutator-relation $(s, t) = t^{p^{\tau-\kappa}}$ with $\kappa \leqq \sigma$, and the elements of the direct Abelian factor have at most the order $p^{\tau-\kappa}$. 4. Conversely, every group generated by two elements s and t satisfying the above relation is quasi-hamiltonian, except when $p=2$ and $\kappa = \tau - 1$. In particular, two elements of order 4 with $s^{-1}ts = t^{-1}$ and $s^2 \neq t^2$ generate a group which is not quasi-hamiltonian. For $s^2 = t^2$ one obtains the quaternion group.

K. A. Hirsch (London).

√ ***MacLane, Saunders. Cohomology theory of Abelian groups.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 8–14. Amer. Math. Soc., Providence, R. I., 1952.

Exposé de résultats concernant l'homologie et la cohomologie "abélienne" d'un groupe (abélien) Π; certains ayant déjà fait l'objet de deux notes d'Eilenberg et MacLane [Proc. Nat. Acad. Sci. U. S. A. 36, 443–447, 657–663 (1950); ces Rev. 12, 350, 520], nous renvoyons à l'analyse de ces notes. Un "complexe abélien" $A(\Pi)$ est construit de proche en proche: $A^0(\Pi)$ est le complexe classique (sans opérateurs) donnant l'homologie et la cohomologie du groupe discret Π; $A^{n+1}(\Pi)$ se déduit de $A^n(\Pi)$ par l'adjonction de cellules qui ont pour effet de transformer en bords les "cycles génériques" de $A^n(\Pi)$; le complexe $A(\Pi)$ est la réunion des $A^n(\Pi)$. Aucune mention n'est faite de l'isomorphisme des groupes $H^p(A^n(\Pi), G)$, resp. $H_p(A^n(\Pi), G)$, avec les "groupes d'Eilenberg-MacLane" $H^{n+p}(\Pi, n+1, G)$, resp. $H_{n+p}(\Pi, n+1, G)$: conjecture démontrée ultérieurement par Eilenberg et MacLane dans une note III [ibid. 37, 307–310 (1951); ces Rev. 13, 151]. Pour terminer, des méthodes de calcul (par générateurs et relations) sont données pour certains groupes, qui sont en fait les suivants: $H^4(\Pi, 2, G)$; $H^{n+2}(\Pi, n, G) \approx \mathrm{Hom}\,(\Pi/2\Pi, G)$ pour $n \geqq 3$; $H_4(\Pi, 2)$ qui n'est autre que le groupe $\Gamma(\Pi)$ de J. H. C. Whitehead [voir ces mêmes Proceedings, vol. 2, pp. 354–357; ces Rev. 13, 575]; $H_5(\Pi, 2)$; $H_6(\Pi, 3) \approx {}_2\Pi + \Lambda(\Pi)$ (où $\Lambda(\Pi) = \Pi \wedge \Pi$, puissance extérieure deuxième du groupe abélien Π); $H_7(\Pi, 4) \approx {}_2\Pi$. *H. Cartan* (Paris).

Cobbe, Anne P. Some algebraic properties of crossed modules. Quart. J. Math., Oxford Ser. (2) 2, 269–285 (1951).

Let $\theta: Q \to A(X)/I(X)$ be a Q-kernel, and G a subgroup of the center Z of X. Let $\beta: X \to \bar{X} = X/G$ and $h: A(X) \to A(X)/I(X)$ be the natural homomorphisms. A crossed module related to (θ, G) is defined essentially as a group Φ containing $\bar{X}$ as a subgroup, together with a homomorphism $\tau: \Phi \to A(X)$, such that: (1) the homomorphisms $\beta: X \to \Phi$, $\tau: \Phi \to A(X)$ constitute a crossed module; (2) there exists a homomorphism $\rho: \Phi \to Q$ such that $\theta\rho = h\tau$ and such that the sequence

$$0 \to G \to X \xrightarrow{\beta} \Phi \xrightarrow{\rho} Q \to 0$$

is exact. [The condition $\rho^{-1}(0) = \bar{X}$, included here, is inadvertently omitted in the definition on p. 271.]

The author offers a classification of the "strict isomorphism classes" of crossed modules related to a given (θ, G). However, the proof of Lemma 5, and hence of Theorem 2A, is unconvincing.

Given an extendible Q-kernel $\theta: Q \to A(X)/I(X)$, let $\mathcal{S}$ denote the set of all equivalence classes of its extensions, and let A_θ denote the group of all automorphisms η of Q such that $\theta\eta = \theta$. Given an extension

$$0 \to X \to E \xrightarrow{\rho} Q \to 0$$

and an element η of A_θ, we obtain another extension

$$0 \to X \to E \xrightarrow{\eta\rho} Q \to 0.$$

In this way A_θ operates as a transformation group on $\mathcal{S}$. A_θ also acts as a group of operators on $H^3(Q, Z)$. Lemma 5 asserts that the orbits in $\mathcal{S}$ under A_θ are in one-to-one correspondence with the orbits in $H^2(Q, Z)$ under A_θ; a footnote adds, "This correspondence is analogous to that between equivalence classes [of extensions] and the elements of $H^2(Q, Z)$." This seems to mean either: (A) each of the canonical one-to-one correspondences between $\mathcal{S}$ and $H^2(Q, Z)$ induces a one-to-one correspondence between the orbits, or (B) there always exists at least one canonical map which does this. (A) is equivalent to the proposition that A_θ always operates trivially on $\mathcal{S}$; this is demonstrably false, as the author recognizes on an earlier page. (B) is equivalent to the proposition that $\mathcal{S}$ always contains a fixed point under A_θ; the reviewer does not know whether this is true or false.

The difficulty can be avoided by considering as the type of object to be classified, not a pair (Φ, τ) for which there exists ρ as above, but a pair (Φ, τ) together with a definite $\rho: \Phi \to Q$ satisfying the conditions mentioned.

R. L. Taylor (New York, N. Y.).

Mills, W. H. Multiple holomorphs of finitely generated abelian groups. Trans. Amer. Math. Soc. 71, 379–392 (1951).

The author determines all cases in which two or more finitely generated Abelian groups have the same holomorph. His results sharpen those previously obtained by G. A. Miller [Math. Ann. 66, 133–142 (1909)]. Theorem I. If each of two finitely generated Abelian groups is isomorphic to a normal subgroup of the holomorph of the other, then the groups are isomorphic. In particular: Two finitely generated Abelian groups with isomorphic holomorphs are isomorphic. (This is no longer true for non-Abelian groups: the dihedral and the dicyclic groups of order $4n$ with $n\geqq 3$ have the same holomorph.) Theorem II. The holomorph H of a finitely generated Abelian group G is also the holomorph of any normal maximal Abelian subgroup of it, isomorphic to G. (For finite Abelian groups the word "maximal" may be omitted.) Theorem III combines theorems I and II and gives therefore a necessary and sufficient condition. There are at most four maximal Abelian normal subgroups of H, isomorphic to G. If G contains no elements of even order or has at least three independent generators of infinite order, then G is the only maximal Abelian normal subgroup of H. The discussion is elementary throughout, but has to use a great number of case distinctions. *K. A. Hirsch.*

Frame, J. S. An irreducible representation extracted from two permutation groups. Ann. of Math. (2) 55, 85–100 (1952).

A group G has two subgroups A, B such that every element of G belongs either to the double coset $G_1=AB$ or else to the double coset $G_2=Ag_2B$ where g_2 is an element of G not in G_1. It follows that the compound characters of G corresponding to the permutation representations generated by A, B respectively possess in common exactly two simple characters, the character which is unity for every element and a second character of degree f. Methods are given for determining this second character and the corresponding irreducible representation F. These results are deduced from the properties of an incidence matrix defined as follows: If $A\rho_i$ are the m right cosets of A in G, and $B\sigma_j$ are the n right cosets of B in G, then $A\rho_i$ and $B\sigma_j$ are said to be incident if they have a common element of G. An $m\times n$ incidence matrix $V=(v_{ij})$ is defined such that $v_{ij}=1$ or 0 according as $A\rho_i$ and $B\sigma_j$ are or are not incident.

Notation. p and $q=1-p$ denote the fractions of the elements of G which belong to G_1 and G_2 respectively; p_λ and q_λ denote the fractions of the elements of G in a class C_λ which belong to G_1 and G_2 respectively; σ is the standard deviation of these p_i's averaged over the group elements; p_{21} is the fraction of the elements of G_2 whose inverses belong to G_1; k_{ir} is the correlation between the incidence numbers in the ith and rth rows of V; $K=[k_{ri}]$ is a correlation matrix the mean square of whose elements is k^2.

The author gets the following expressions for the degree f of the irreducible representation F,

$$f=1/k^2=q(p-p_{21}).$$

The characters of the representation are obtained in the alternative forms

$$\chi_\lambda=f(1-q_\lambda/q)=(p_\lambda-p)/(p-p_{21})=(p_\lambda-p)/\sigma.$$

A method is given for obtaining the actual matrix representation. The general method depends on the fact that if $\bar{A}$, $\bar{B}$, $\bar{G}$, etc., denote the arithmetic means of the elements in the respective groups or cosets, then $(fp/q)(\bar{A}\bar{B}-\bar{G})$ is a primitive idempotent belonging to the required representation.

Two examples are given. The first is a simple example concerning the octahedral group. The second concerns the group of order 51, 840 associated with the automorphisms of the 27 lines on a cubic surface. The author has succeeded (he says) in finding explicitly 3 irreducible representations of this group, but the details are too lengthy to be included in the present paper. *D. E. Littlewood* (Bangor).

√★**Brauer, Richard. On the representations of groups of finite order.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 33–36. Amer. Math. Soc., Providence, R. I., 1952.

A function χ on a finite group G of order n is an (absolutely) irreducible character of G if and only if: I) the restriction of χ to an elementary subgroup H is a character of H (where a group is called elementary when it is a direct product of a cyclic group and a p-group), II) χ is a class function, and III) $(1/n)\sum|\chi(g)|^2=1$. Provided that $\chi(1)>0$, I) may be replaced by the statement that the restriction to H is a linear combination of characters of H. A still different set of necessary and sufficient conditions is obtained by replacing I) by a certain condition dealing with a subgroup and its cyclic factor group, and these sets combined prove to be powerful in the investigation of group characters. They also give a necessary and sufficient condition for the solubility of a group. The theory of modular representations furnishes further theorems concerning the relationship of the representations of G with those of its suitable subgroups [Brauer, Proc. Nat. Acad. Sci. U. S. A. **30**, 109–114 (1944); **32**, 182–186, 215–219 (1946); these Rev. **6**, 34; **8**, 14, 131].

A theorem is given which has the effect of reducing the problem of the Schur index of an irreducible character of G to those of the characters of semi-elementary subgroups of G, where a group H is called semi-elementary if H possesses a cyclic normal subgroup N such that H/N is a p-group; the theorem is a refinement of an earlier result of the author [Ann. of Math. (2) **48**, 502–514 (1947); these Rev. **8**, 503]. The problem can further be reduced to the case of a group H possessing a cyclic normal subgroup N such that H/N is an abelian p-group. The result leads to a determination of the splitting field of an irreducible representation, covers the author's former proof of Schur's conjecture, and gives further results in this context. *T. Nakayama* (Nagoya).

Robinson, G. de B. On the modular representations of the symmetric group. Proc. Nat. Acad. Sci. U. S. A. **37**, 694–696 (1951).

It is shown, on making use of the former results concerning the block distribution [Brauer, Trans. Roy. Soc. Canada. Sect. III. (3) **41**, 11–19 (1947); Robinson, ibid. **41**, 20–25 (1947); Nakayama and Osima, Nagoya Math. J. **2**, 111–117 (1951); Staal, Canadian J. Math. **2**, 79–92 (1950); these Rev. **10**, 678; **12**, 672; **11**, 415], that the numbers of irreducible ordinary, resp. modular, representations in a block of a symmetric group depend only on the defect of the block, which is in turn determined uniquely by the number of removable p-hooks of the corresponding diagram. The ordinary representation case settles a conjecture by Chung [ibid. **3**, 309–327 (1951); these Rev. **13**, 106]. It is observed that the star diagrams for composite hook lengths have applications for the structure of decomposition matrices. *T. Nakayama* (Nagoya).

Putnam, Calvin R., and Wintner, Aurel. The orthogonal group in Hilbert space. Amer. J. Math. **74**, 52–78 (1952).

A number of facts are obtained which show that the orthogonal group G on a real $\aleph_0$-dimensional Hilbert space behaves differently from its finite-dimensional counterpart. Let G_0 be the set of all (orthogonal) operators of the form e^S where S is a bounded and skew-symmetric linear operator, and let G_1 be the subset of G consisting of operators A such that $\|A-I\|=2$ (I=identity operator, $\|\cdot\|$ is the operator bound). Among other results: a) G_0 is not closed under multiplication; b) an element of $G-G_0$ may be the product of two such elements (these are reflections in the finite-dimensional case); c) G_1 and $G-G_1$ are both arcwise connected. *I. E. Segal* (Chicago, Ill.).

Dieudonné, Jean. Sur les systèmes maximaux d'involutions conjuguées et permutables dans les groupes projectifs. Summa Brasil. Math. **2**, no. 6, 59–94 (1950).

Let K and K' be division rings. In a recent study of the automorphisms of the classical groups [see the second following review] the author has shown among other things that when n and m are integers greater than or equal to 5 then the projective groups $PSL_n(K)$ and $PSL_m(K')$ need not be isomorphic unless $n=m$ and K is isomorphic either to K' or to the transpose of K'. In the present paper he extends this result by showing that the conclusion holds for all pairs of positive integers n, m except possibly for $n=m=2$, $n=m=4$, and $n=2$, $m=3$. The proof depends upon determining explicitly the number of elements in certain families of involutions in groups associated with $PSL_n(K)$. The results of these determinations are summarized in the first six of the seven theorems of the paper. The computations are long and complicated and involve a division into cases according to the nature of K and the integers n and m. *G. W. Mackey* (Cambridge, Mass.).

Dieudonné, Jean. On the orthogonal groups over the rational field. Ann. of Math. (2) **54**, 85–93 (1951).
Dieudonné, Jean. Orthogonal and unitary groups over the rational field. Amer. J. Math. **73**, 940–948 (1951).

Let E be an n-dimensional space over the field Q of rational numbers and let f be a non-degenerate symmetric bilinear form defined on E with $g(x)=f(x,x)$ the corresponding quadratic form. The index of f is denoted by ν. [Cf. Dieudonné, Sur les groupes classiques, Actualités Sci. Ind., no. 1040, Hermann, Paris, 1948; these Rev. **9**, 494]. The corresponding orthogonal group is denoted by $O_n(Q,f)$ and its subgroup of rotations by $O_n^+(Q,f)$. The commutator subgroup of $O_n(Q,f)$ is denoted by $\Omega_n(Q,f)$ and the corresponding projective group by $P\Omega_n(Q,f)$. From a previous result of the author [loc. cit., p. 29], in case $n\geqq 5$ and $\nu\geqq 1$ the group $P\Omega_n(Q,f)$ is simple. By a theorem of A. Meyer [Vierteljschr. Naturforsch. Ges. Zürich **29**, 209–222 (1884)], if g is indefinite and $n\geqq 5$, then $\nu\geqq 1$. The main result in the first of the papers being reviewed here is that $P\Omega_n(Q,f)$ is also simple for any positive definite g and $n\geqq 6$. (The case $n=6$ is dealt with in the second paper.) The following automorphism theorems, which supplement results obtained by the author in the paper reviewed below, are consequences of the above theorem: For $n\geqq 8$ and $\nu=0$, every automorphism of $O_n(Q,f)$ is of the form $u\to\chi(u)huh^{-1}$, where χ is a representation of $O_n(Q,f)$ in $\{-1,+1\}$ and h is a linear mapping of E onto E such that $g(h(x))=\lambda g(x)$; every automorphism of $PO_n(Q,f)$ is induced by an automorphism of $O_n(Q,f)$. For n even $\geqq 10$ and $\nu=0$, every automorphism on $O_n^+(Q,f)$ is induced by an automorphism of $O_n(Q,f)$; every automorphism of $PO_n^+(Q,f)$ is induced by an automorphism of $O_n^+(Q,f)$; every automorphism of $P\Omega_n(Q,f)$ is induced by an automorphism of $O_n(Q,f)$. It is also shown that the group $O_n(Q,f)/\Omega_n(Q,f)$ is infinite for any f and $n\geqq 2$. The automorphism result for the orthogonal groups has also been established by the reviewer, using very different methods, for arbitrary fields and dimensions (even infinite) $n\geqq 5$, if $\nu\neq 0$, and $n\geqq 3$, if $\nu=0$.

In the second paper, the author considers any quadratic extension K_1 of Q, an n-dimensional space E_1 over K_1 and a non-degenerate Hermitian form f on E_1. He then proves that the commutator subgroup of the projective unitary group $PU_n^+(K_1,f_1)$ is always simple for $n\geqq 5$. In view of results obtained in the author's monograph on the classical groups cited above [p. 70], the only case that required proof here was that in which K_1 is an imaginary quadratic field and f_1 is positive definite. *C. E. Rickart.*

Dieudonné, Jean. On the automorphisms of the classical groups. With a supplement by Loo-Keng Hua. Mem. Amer. Math. Soc., no. **2**, vi+122 pp. (1951). $1.80.

The author here makes a systematic attack on the problem of determining the automorphisms of the classical groups over arbitrary sfields (division rings). Essentially the only previous results beyond the classical case were obtained by O. Schreier and B. L. van der Waerden [Abh. Math. Sem. Univ. Hamburg **6**, 303–322 (1928)] who determined the automorphisms of the unimodular projective group $PSL_n(K)$ ($n>1$) over a commutative field K. Their results give without difficulty the automorphisms of the general linear group $GL_n(K)$ and the unimodular group $SL_n(K)$ for $n>1$ and commutative K. However, the methods used by these authors depend essentially on the commutativity of K. The general method used in the present paper consists in considering the manner in which the involutions in the groups are transformed by the automorphisms. This method, which goes back to E. Cartan, was also used by G. Mackey [Ann. of Math. **43**, 244–260 (1942); these Rev. **4**, 12] in studying infinite-dimensional normed vector spaces. Some of the techniques introduced by Mackey are used to good advantage here. The author also relies heavily on the structure theory developed for these groups in his monograph on the subject [Actualités Sci. Ind., no. 1040, Hermann, Paris, 1948; these Rev. **9**, 494], a fact which accounts for certain of the restrictions needed in some of the theorems. Needless to say, the treatment given here applies also in the commutative case. In the supplement, L. K. Hua disposes of certain of the lowest dimensional cases which were left open by Dieudonné. Hua's methods involve direct computation with matrices and are quite effective for the low dimensions where the other methods fail.

We summarize below the main results obtained in the memoir (including the supplement). The n-dimensional vector space on which the linear transformations act will be denoted by E and its dual by E^*. The sfield of scalars is denoted by K and its multiplicative group of non-zero elements by K^*. If f is a fundamental bilinear form on E, $f(x,y)$ is abbreviated to (x,y) and the index of f [cf. Dieudonné, loc. cit., p. 17] is denoted by ν. Notations from the author's monograph cited above are used freely.

1. Every automorphism of $GL_n(K)$ ($n\geqq 2$) takes one of the two forms $u\to\chi(u)gug^{-1}$ or $u\to\chi(u)h\check{u}h^{-1}$, where $u\to\chi(u)$ is a representation of $GL_n(K)$ in the multiplicative group of the center of K, g is a semi-linear transformation of E onto

E; h is a semi-linear transformation of E onto E^* and $\check{u}$ denotes the transformation contragredient to u. Every automorphism of $PGL_n(K)$ ($n\geqq 2$) is induced by an automorphism of $GL_n(K)$.

2. Every automorphism of $SL_n(K)$ ($n\geqq 2$) is the restriction of an automorphism of $GL_n(K)$, with the possible exception of the cases $n=2$ or 4 when K is non-commutative, has characteristic different from 2 and is such that -1 is not in the commutator subgroup of K^*. Every automorphism of $PSL_n(K)$ (same restrictions as above) is induced by an automorphism of $SL_n(K)$.

3. Every automorphism of $Sp_{2m}(K)$ ($m\geqq 2$ and $m\neq 2$ if K has only two elements) is of the form $u\to gug^{-1}$, where g is a semi-linear transformation of E onto E and $(g(x), g(y))=\lambda(x, y)^\sigma$, where $\lambda\varepsilon K$ and σ is the automorphism of K associated with g. Every automorphism of $PSp_{2m}(K)$ (same restrictions as above) is induced by an automorphism of $Sp_{2m}(K)$.

4. Every automorphism of $O_n(K, f)$ ($n\geqq 4$, K a field with characteristic different from 2, index $\nu\geqq 1$) is of the form $u\to\chi(u)gug^{-1}$, where $u\to\chi(u)$ is a representation of $O_n(K, f)$ in the multiplicative group $\{1, -1\}$, g is a semi-linear transformation of E onto E and $(g(x), g(y))=\lambda(x, y)^\sigma$, where $\lambda\varepsilon K$ and σ is the automorphism of K associated with g. (This result is also true for $n=3$, $\nu=1$ and $n=3$, $\nu=0$.) For n even, every automorphism of $PO_n(K, f)$ is induced by an automorphism of $O_n(K, f)$. For n odd, $PO_n(K, f)=PO_n^+(K, f)$ and is isomorphic to $O_n^+(K, f)$.

5. Every automorphism of $O_n^+(K, f)$ (same restrictions as in 4 with $n\geqq 5$) is the restriction of an automorphism of $O_n(K, f)$. There are automorphisms of $O_4^+(K, f)$ ($\nu=2$) which are not restrictions of automorphisms of $O_4(K, f)$. (Cf. Hua's supplement.) For even $n\geqq 6$ and $\neq 8$, every automorphism of $PO_n^+(K, f)$ (same restrictions as above) is induced by an automorphism of $O_n^+(K, f)$. When $n=8$, exceptions actually occur only when $\nu=4$ and -1 is a square in K or when -1 is not a square in K.

6. Let K be a field which is a separable extension of degree 2 over a field K_0 with characteristic different from 2. Let $\xi\to\bar{\xi}$ be the automorphism of K over K_0 distinct from the identity and denote by N the multiplicative group of elements of K of norm 1. Consider $U_n(K, f)$, where f is a form in E Hermitian with respect to $\xi\to\bar{\xi}$.

Every automorphism of $U_n(K, f)$ ($n\geqq 3$, $\nu\geqq 1$), with possible exception of $U_3(F_9)$ and $U_3(F_{25})$, is of the form $u\to\chi(u)gug^{-1}$ where $u\to\chi(u)$ is a representation of $U_n(K, f)$ in N, g is a semi-linear transformation of E onto E such that $(g(x), g(y))=\lambda(x, y)^\sigma$, $\lambda\varepsilon K_0$ and σ is the automorphism of K associated with g.

This same result holds when K is a reflexive sfield except then $U_n(K, f)$ coincides with its commutator subgroup so that $\chi(u)\equiv 1$. In the first case, when $n\neq 4$, every automorphism of $U_n^+(K, f)$ is the restriction of an automorphism of $U_n(K, f)$. For $n\geqq 3$, $n\neq 4$ and K a finite field with characteristic different from 2, every automorphism of $PU_n^+(K)$, with possible exception of $PU_3(F_9)$ and $PU_3(F_{25})$, is induced by an automorphism of $U_n^+(K)$.

7. If K is a finite field with characteristic equal to 2, then every automorphism of $\Omega_n(K, f)$ ($n\geqq 10$, $\nu\geqq 1$) is of the form $u\to gug^{-1}$, where g is a semi-linear transformation of E onto E such that $N(g(x))=\lambda(N(x))^\sigma$, σ is the automorphism of K associated with g and $N(x)$ is the quadratic form which is associated with f in this case [cf. Dieudonné, loc. cit., p. 40]. Note that $P\Omega_n(K, f)=\Omega_n(K, f)$.

If K is a finite field with characteristic different from 2, then every automorphism of $P\Omega_n(K, f)$ ($n\geqq 6$, $n\neq 8$) is induced by an automorphism of $O_n^+(K, f)$.

8. In all but a few exceptional cases the automorphisms of the simple finite groups derived from the classical groups [Dickson, Linear groups . . . , Teubner, Leipzig, 1901] are determined. In each of these cases the Schreier hypothesis is seen to hold; that is, if A is the group of automorphisms of G, then A/G is solvable. Furthermore, except for those isomorphisms already listed by Dickson [loc. cit.], no two of these groups are isomorphic.

9. For K with characteristic different from 2, the above results for $GL_n(K)$, $Sp_{2m}(K)$, $O_n(K, f)$ and $U_n(K, f)$ have been extended by the reviewer to infinite dimensions [Amer. J. Math. 72, 451–464 (1950); these Rev. 11, 729; and the two papers reviewed below]. The methods used, which are quite different from those used here, also apply for finite dimensions at least equal to 3, in the case of $GL_n(K)$, and at least 6 in the other cases. In addition, the index restriction $\nu\geqq 1$ is removed in the last two cases and, in the last case, K can be any division ring with an involution $\xi\to\bar{\xi}$.

C. E. Rickart (New Haven, Conn.).

Rickart, C. E. Isomorphic groups of linear transformations. II. Amer. J. Math. 73, 697–716 (1951).

L'auteur détermine dans ce travail les automorphismes du groupe symplectique $Sp_n(K)$ et des groups unitaires $U_n(K, f)$; dans ces derniers, K est un corps commutatif ou non, de caractéristique $\neq 2$, ayant plus de 3 éléments, et muni d'une involution; le groupe orthogonal est inclus comme cas particulier. La dimension n est toujours supposée $\geqq 6$, mais peut être infinie; en outre, les méthodes de l'auteur lui donnent aussi tous les isomorphismes d'un $U_n(K, f)$ sur un $U_m(K', f')$ (et l'analogue pour les groupes symplectiques, ou pour le cas de la dimension infinie). Les progrès par rapport aux résultats du rapporteur [voir l'analyse ci-dessus] sont considérables: tous les groupes unitaires et orthogonaux sont traités d'un seul coup, le corps K n'est pas supposé réflexif, on a à la fois automorphismes et isomorphismes, et enfin la forme f peut être d'indice 0. Ces progrès sont dûs à une caractérisation très simple des involutions minimales (i.e. dont un des sous-espaces a la plus petite dimension) dans les groupes considérés; l'idée est la même que dans le travail antérieur de l'auteur sur les automorphismes du groupe linéaire général [Amer. J. Math. 72, 451–464 (1950); ces Rev. 11, 729], et se rattache à la technique des "couples minimaux" de Mackey: pour tout ensemble S d'involutions, soit $c(S)$ l'ensemble des involutions qui commutent avec tous les éléments de S. On remarque alors que si, pour deux involutions T_1, T_2, $\rho(T_1, T_2)$ désigne le nombre d'éléments de $c(c(T_1, T_2))$, le maximum de $\rho(T_1, T_2)$ lorsque T_2 parcourt toutes les involutions commutant avec T_1, a une même valeur ρ lorsque T_1 n'est pas minimale, et la valeur $\frac{1}{2}\rho$ lorsque T_1 est minimale. Cela fait, le reste de la démonstration procède suivant la méthode habituelle (utilisation du théorème fondamental de la géométrie projective). Il serait intéressant d'étendre la méthode aux groupes $O_n^+(K, f)$ et $U_n^+(K, f)$ (K commutatif), ainsi qu'aux groupes projectifs. *J. Dieudonné.*

Rickart, C. E. Isomorphisms of infinite-dimensional analogues of the classical groups. Bull. Amer. Math. Soc. 57, 435–448 (1951).

Article d'exposition où l'auteur, après avoir donné des définitions très générales des groupes classiques, valables pour des espaces de dimension infinie sur des corps non

commutatifs, esquisse les grandes lignes de la méthode qu'il a développée dans deux travaux récents [Amer. J. Math. **72**, 451–464 (1950); ces Rev. **11**, 729; et l'oeuvre analysé ci-dessus] pour déterminer les automorphismes de ces groupes, et plus généralement, leurs isomorphismes les uns sur les autres. Les résultats des deux articles précités sont légèrement généralisés au cas où les groupes considérés peuvent contenir des transformations semi-linéaires, les méthodes restant tout à fait analogues, avec quelques légères modifications. *J. Dieudonné* (Nancy).

Galbură, Gh. Variétés-groupe de dimension 3. An. Acad. Repub. Pop. Române. Ser. Mat. Fiz. Chim. **3**, no. 18, 428–438+ii (1950). (Romanian. Russian and French summaries)

By considering their normal subgroups, the author obtains an enumeration of the connected 3-dimensional Lie groups. The topology of each group is described. *P. A. Smith* (New York, N. Y.).

Samelson, Hans. Topology of Lie groups. Bull. Amer. Math. Soc. **58**, 2–37 (1952).

Dans cette conférence (30 pages de texte, 6 pages de bibliographie), l'auteur passe une revue complète des résultats concernant l'homologie et l'homotopie des groupes de Lie et de leurs espaces homogènes; c'est le premier exposé d'ensemble depuis celui de E. Cartan en 1936; il rend compte des progrès les plus récents, et même de résultats encore inédits. Voici un bref sommaire: rappel des notions fondamentales classiques sur les algèbres de Lie, la classification des groupes simples, le groupe compact associé à un groupe semi-simple. Topologie des groupes résolubles (Chevalley).—Groupes compacts: complète réductibilité de l'algèbre de Lie, extensions de groupes compacts, théorème de Weyl (le groupe fondamental d'un G compact semi-simple est fini), structure des groupes compacts.—Nombres de Betti des groupes de Lie (compacts): résultats explicites (Pontrjagin); méthodes générales (théorème de Hopf, méthode des formes différentielles biinvariantes, méthodes algébriques: Chevalley-Eilenberg, Koszul). Nombres de Betti des groupes exceptionnels (Yen, Chevalley). Torsion des groupes de Lie (Pontrjagin, A. Borel).—Espaces homogènes de groupes compacts: caractéristique d'Euler-Poincaré; théorème de Montgomery (si E compact est espace homogène d'un G avec groupe d'isotropie connexe, E est espace homogène d'un groupe compact).—Diagramme d'un groupe compact: le groupe Φ de Weyl dans un tore maximal, équivalence des définitions du rang.—Résultats généraux sur la cohomologie des espaces homogènes: théorème de Samelson, résultats de Leray, méthode algébrique de Koszul, rôle de la transgression et de l'algèbre de Weil d'une algèbre de Lie; résultats récents de Borel sur la torsion des espaces homogènes. Quelques indications sur la topologie des espaces homogènes non compacts.—Homotopie des groupes de Lie et questions connexes (cette partie est assez peu développée).—Questions particulières: grassmanniennes, espaces riemanniens symétriques de E. Cartan, détermination des sous-groupes de rang maximum des groupes simples compacts, des espaces homogènes de dimension 2, prolongement d'un germe d'espace homogène (condition pour qu'une sous-algèbre de l'algèbre de Lie de G simplement connexe engendre un sous-groupe fermé dans G: Mal'cev, Mostov), détermination des groupes opérant transitivement dans les sphères, etc. *H. Cartan* (Paris).

Harish-Chandra. Plancherel formula for complex semi-simple Lie groups. Proc. Nat. Acad. Sci. U. S. A. **37**, 813–818 (1951).

Let G be a connected semi-simple Lie group. The author outlines his derivation of the explicit generalized Peter-Weyl-Plancherel formula for $L_2(G)$. A very lengthy and difficult derivation of such a formula has been published by Gel'fand and Naĭmark [Izvestia Akad. Nauk SSSR. Ser. Mat. **11**, 411–504 (1947); these Rev. **9**, 495] for the case where G is the group G_2 of all complex 2×2 matrices of determinant 1 and they also outlined a proof for the $n\times n$ case [Doklady Akad. Nauk SSSR (N.S.) **63**, 609–612 (1948); these Rev. **10**, 429]. The present approach differs considerably from that of Gel'fand and Naĭmark. It is of a somewhat more algebraic nature and relies very much less on lengthy computations, but is based on the general theory of semi-simple Lie algebras and Lie groups. Especially, when applied to the above group G_2 it leads to a proof which is much simpler than that of Gel'fand and Naĭmark.

Although the existence of such a Plancherel formula has of course been known now for a few years for arbitrary locally compact separable unimodular groups, its actual explicit determination appears to present new difficulties which vary considerably with the kind of group in question. Thus no explicit formula is as yet known for real semi-simple Lie groups, but the author has recently informed the reviewer that he has obtained one for the group of real 2×2 matrices of determinant 1. *F. I. Mautner.*

Nakayama, Tadasi. Remark on the duality for noncommutative compact groups. Proc. Amer. Math. Soc. **2**, 849–854 (1951).

L'auteur montre comment, dans la dualité de Tannaka pour un groupe compact G, on peut associer un "objet dual" à tout sousgroupe de G. Dans chaque classe d'équivalence $\mathfrak{D}_k$ de représentations continues de G, soit $x\to M_k(x)$ une représentation, $\mathfrak{R}_k$ l'anneau (pour la convolution) des combinaisons linéaires des coefficients de $M_k(x)$, $\mathfrak{R}$ la somme de tous les anneaux $\mathfrak{R}_k$ (sous-anneau de l'anneau R des fonctions complexes continues dans G); on sait que $\mathfrak{R}$ est aussi un anneau pour la multiplication ordinaire, et est invariant par conjugaison. En outre, R, $\mathfrak{R}$ et $\mathfrak{R}_k$ sont des G-modules à gauche pour la multiplication $(s\cdot f)(x)=f(s^{-1}x)$. Cela étant, à un sous-groupe fermé H de G on fait correspondre le sous-anneau $\mathfrak{R}'=\mathfrak{R}\cap R'$ où R' est l'anneau des fonctions continues f telles que $f(xa)=f(x)$ pour $a\in H$; $\mathfrak{R}'$ est somme directe des $\mathfrak{R}_k'=\mathfrak{R}_k\cap R'$; $\mathfrak{R}_k'$ est somme directe de l_k G-modules à gauche irréductibles, dont on peut prendre les bases sous forme des colonnes d'une matrice $N_k(x)$, qu'on démontre être de la forme $N_k(x)=M_k(x)T_k$, T_k étant une matrice indépendante de $x\in G$. On démontre les formules

$$(1)\qquad N_k(x)\otimes N_h(x)=P_{kh}\begin{pmatrix} N_{\omega_1(k,h)}(x) & & \\ & N_{\omega_2(k,h)}(x) & \\ & & \ddots \end{pmatrix}B_{kh},$$

$$\bar{N}_k(x)=U_kN_{m(k)}(x)C_k,$$

où U_k et P_{kh} (indépendantes de x) ne dépendent que des représentations $x_k\to M_k(x)$ (ainsi que les fonctions $k\to m(k)$, $(k, h)\to\omega_i(k, h)$), B_{kh} et C_k (indépendantes de x) dépendent de H. Inversement, si on se donne un système de matrices T_k, C_k, B_{kh} telles que les $N_k(x)=M_k(x)T_k$ satisfassent à (1), l'auteur prouve qu'il lui correspond le sous-groupe fermé H des $a\in G$ tels que $T_k=N_k(a)$ pour tout k, de sorte que les fonctions $f\in\mathfrak{R}$ telles que $f(xa)=f(x)$ pour $a\in H$ soient les combinaisons linéaires des éléments des $N_k(x)$. On a aussi

une généralisation du théorème de Tannaka caractérisant les $x \epsilon G$ comme donnant les homomorphismes $f \to f(x)$ de $\mathfrak{R}$ dans le corps des complexes. Lorsque G est abélien, on retrouve les résultats classiques de la théorie de Pontrjagin.

J. Dieudonné (Nancy).

Areškin, G. Ya. Operator structures of locally compact topological groups with a countable basis. Doklady Akad. Nauk SSSR (N.S.) **81**, 129–132 (1951). (Russian)

The author applies here his earlier work on lattices of closed sets in various classes of topological spaces [same Doklady **59**, 629–630 (1948); Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 213–220 (1949); these Rev. **9**, 455; **10**, 726]. He begins by characterizing lattices isomorphic to bases for closed sets in locally compact regular spaces, being presumably unaware that this result is trivially contained in his previous ones, since a locally compact Hausdorff space is necessarily regular. He next considers the lattice properties of a basis for closed sets in a locally compact topological group G satisfying the second countability axiom. Noting that the arithmetic product $P \cdot Q$ of two compact subsets P and Q of G is again compact, he sets down axioms for a lattice with a multiplication operation, and by adjoining further axioms, to a total of 12, he succeeds in characterizing lattices with operators which are isomorphic to bases for closed sets (with certain additional properties, such as containing the unit element of G) for locally compact topological groups with the second countability axiom. No proofs are given.

E. Hewitt (Seattle, Wash.).

Segal, I. E. A class of operator algebras which are determined by groups. Duke Math. J. **18**, 221–265 (1951).

Soit $x \to U_x$ une représentation unitaire d'un groupe analytique G sur un espace de Hilbert $\mathfrak{H}$; on pose $U_f = \int U_x f(x) dx$ pour f sommable par rapport à dx (mesure invariante à gauche), et on note $\mathfrak{D}$ le sous-espace (invariant et partout dense) engendré par les $U_f \mathbf{a}$, où f est indéfiniment différentiable et à support compact. Si X est un opérateur différentiel sur G, invariant à droite, il existe [Gårding, Proc. Nat. Acad. Sci. U. S. A. **33**, 331–332 (1947); ces Rev. **9**, 133] un opérateur (en général non borné) $\bar{X}$ défini sur $\mathfrak{D}$ tel que $\bar{X} U_f \mathbf{a} = U_{Xf} \mathbf{a}$ pour f indéfiniment différentiable à support compact. L'auteur démontre que si X est un polynôme selfadjoint de degré $\leqq 2$ par rapport aux éléments de l'algèbre de Lie de G, alors $\bar{X}$ est "essentiellement hypermaximal" (i.e. $\bar{X}^* = \bar{X}^{**}$) si G est unimodulaire; donc si X commute à l'algèbre de Lie et si la représentation est irréductible alors $\bar{X}$ est un scalaire. [Note du rapporteur: l'hypothèse que degré de $X \leqq 2$ est évidemment superflue; il suffit d'observer que $\bar{X}$ applique $\mathfrak{D}$ dans $\mathfrak{D}$ ainsi que $\bar{X}^*$, en sorte qu'on a le droit d'écrire $4\bar{X} = (\bar{X}+1)(\bar{X}+1)^* - \cdots$, formule qui montre que $\bar{X}$ est combinaison d'opérateurs symétriques positifs sur $\mathfrak{D}$, lesquels commutent aux U_x si X est dans le centre de l'algèbre associative engendrée par l'algèbre de Lie; mais un opérateur symétrique positif est (Friedrichs) prolongeable en un opérateur selfadjoint, donc a une décomposition spectrale, etc.] L'auteur montre aussi que la représentation de G est engendrée par la représentation $X \to \bar{X}$, i.e. que $x = \exp(X)$ implique $U_x = \exp(\bar{X})$.

L'auteur étudie ensuite les groupes localement compacts G d'homéomorphismes d'un espace localement compact M, avec une mesure invariante μ; dans $L^2(M; \mu)$ on a une représentation unitaire évidente $g \to U_g$ de G; l'auteur démontre (ce qui est pour le moins facile) que μ est ergodique relativement à G si et seulement si l'ensemble des opérateurs U_g et des multiplications par des fonctions continues tendant vers 0 à l'infini est irréductible (la démonstration de l'auteur est compliquée par le fait qu'il n'utilise pas le théorème de Lebesgue-Nikodym, cependant valable même dans les espaces non dénombrables à l'infini; notons aussi que l'invariance de μ n'est pas nécessaire; il suffit que la famille des ensembles de mesure nulle soit invariante par G).

L'auteur étudie ensuite ce qu'il appelle les "mouvements" d'une algèbre autoadjointe uniformément fermée $\mathbf{A}$ d'opérateurs sur un espace de Hilbert; ce sont simplement les groupes à un paramètre de *-automorphismes de $\mathbf{A}$. Si $\mathbf{A}$ est commutative (donc isomorphe à l'algèbre des fonctions continues tendant vers 0 à l'infini d'un espace localement compact Γ) ces mouvements correspondent aux groupes à un paramètre d'homéomorphismes de Γ sur lui-même, et les formes positives sur $\mathbf{A}$ invariantes par un mouvement correspondent, sur Γ, à des mesures positives bornées invariante par les homéomorphismes correspondants. On montre ensuite que si $t \to \xi_t$ est un mouvement de $\mathbf{A}$ (qui n'est plus supposée commutative) et si $A \to U_A$ est la représentation unitaire de $\mathbf{A}$ définie par une forme positive invariante par les ξ_t, alors il existe dans l'espace de cette représentation un groupe à un paramètre $t \to S_t$ d'opérateurs unitaires tel que $U_{\xi_t(A)} = S_t U_A S_t^{-1}$. Tout cela est très facile.

L'article se termine par quelques exemples inspirés de problèmes physiques, à propos desquels l'auteur émet des opinions et suggestions dont la discussion demanderait des connaissances cosmologiques et métaphysiques que le rapporteur n'a malheureusement pas eu le temps d'acquérir.

R. Godement (Nancy).

Hano, Jyun-ichi. On the differentiability of the unitary representation of the Lie group. J. Math. Soc. Japan **2**, 270–283 (1951).

Given a Lie group G and a continuous unitary representation of G in a Hilbert space $\mathfrak{H}$. The main result is that the operators obtained by differentiating along all one-parameter subgroups of G have a common domain which is dense in $\mathfrak{H}$. The author does not mention Gårding's proof of this result [Proc. Nat. Acad. Sci. U. S. A. **33**, 331–332 (1947); these Rev. **9**, 133]. Gårding's proof seems simpler and more elementary. The author deduces that one obtains a representation of the Lie algebra of G in the above dense subspace of $\mathfrak{H}$. In this connection compare F. I. Mautner [Ann. of Math. (2) **52**, 528–556 (1950); these Rev. **12**, 157] and I. E. Segal [the paper reviewed above]. In the course of his argument the author also derives Stone's theorem on one-parameter groups of unitary operators.

F. I. Mautner.

NUMBER THEORY

Mordell, L. J. On the equation $ax^2+by^2-cz^2=0$. Monatsh. Math. **55**, 323–327 (1951).

A short proof is given of Legendre's theorem that the equation $f(x, y, z) = ax^2+by^2-cz^2=0$, where a, b, c are positive square-free integers, relatively prime in pairs, has a solution other than $(0, 0, 0)$ if, and only if, $-ab$ is a quadratic residue of c, bc of a and ca of b. Such a solution is an immediate consequence of the existence of a non-trivial solution of the congruence $f(x, y, z) \equiv 0 \pmod{4abc}$ with $|x| \leqq \sqrt{(2bc)}$, $|y| < \sqrt{(2ac)}$, $z < 2\sqrt{(ab)}$. A solution of this

congruence is obtained with the use of the quadratic reciprocity law from the following elementary result. Let $L_1, L_2, \cdots, L_n$ be homogeneous linear forms with integer coefficients in x, y, z. Let $q_1, q_2, \cdots, q_n$ be positive integers, and r_1, r_2, r_3 positive numbers for which $r_1r_2r_3 \geqq q_1q_2\cdots q_n$. Then a solution of the congruences $L_1 \equiv 0 \pmod{q_1}, \cdots, L_n \equiv 0 \pmod{q_n}$ exists other than $(0, 0, 0)$ with $|x| \leqq r_1$, $|y| < r_2$, $|z| < r_3$. *D. Derry* (Vancouver, B. C.).

Maxfield, J. E. Sums and products of normal numbers. Amer. Math. Monthly **59**, 98 (1952).

Jarden, Dov. On a sequence with separate recurring formulas for members with even and odd subscripts. Riveon Lematematika **5**, 39–40 (1951). (Hebrew. English summary)

The author remarks that a sequence $\{W_n\}$ satisfying the pair of 2nd order recursion relations:

$$W_{2n+2} = aW_{2n+1} + bW_{2n},$$
$$W_{2n+3} = cW_{2n+2} + dW_{2n+1},$$

satisfies the single 4th order recursion relation

$$W_n = (b+ac+d)W_{n-2} - bdW_{n-4}.$$

It is noted that use of this fact would have simplified the work of Tuchman and Kalai [see the following review].
E. G. Straus (Los Angeles, Calif.).

Tuchman, Zevulun, and Kalai, Shraga. Application of recurring sequences for solving Diophantine equations. Riveon Lematematika **5**, 23–31 (1951). (Hebrew. English summary)

The authors prove: If the sequences $\{u_n\}$, $\{v_n\}$ are determined by

$u_0 = 0,\ u_1 = 1,\ u_{2n} = u_{2n-1} + u_{2n-2},\ u_{2n+1} = (k-2)u_{2n} + u_{2n-1}$, and $v_0 = 2,\ v_1 = 1,\ v_{2n} = (k-2)v_{2n-1} + v_{2n-2},\ v_{2n+1} = v_{2n} + v_{2n-1}$, then the pairs $x = u_{2m}$, $y = u_{2m+2n}$ $(m = 0, \pm 1, \pm 2, \cdots)$ yield (except for signs) all integer solutions of the Diophantine equations $x^2 - v_{2n}xy + y^2 = u_{2n}^2$. Similar relations are mentioned for the various other combinations of the u and v with even or odd indices and the corresponding quadratic Diophantine equations. The authors' work would have been greatly simplified by noticing that each of the sequences $\{u_{2n}\}$, $\{u_{2n-1}\}$, $\{v_{2n}\}$, $\{v_{2n-1}\}$ satisfies the difference equation $X_n = kX_{n-1} - X_{n-2}$, as remarked by Jarden [see the preceding review]. *E. G. Straus* (Los Angeles, Calif.).

Buquet, A. Structure en réseau des solutions en nombres rationnels de l'équation diophantienne

$$f(t) \equiv At^4 + Bt^3 + Ct^2 + Dt + E = s^2.$$

Mathesis **60**, 239–243 (1951).

Associated with the given equation are certain quadratic equations with discriminant $f(t)$, and the author extends his previous study [Mathesis **59**, 233–236 (1950); these Rev. **12**, 590] by determining certain algebraic relations between two such auxiliary quadratics. *I. Niven* (Eugene, Ore.).

Sastry, S. On Prouhet-Lehmer problem. J. Sci. Res. Benares Hindu Univ. **1** (1950–1951), 1–4 (1951).

Let $P(k, s)$ denote the least number j such that s sets of j numbers each have equal sums of nth powers for $n = 1, 2, \cdots, k$. The author proves in an elementary way that $P(4, 3) \leqq 8$, $P(4, 4) \leqq 10$, $P(6, 3) \leqq 13$, and in general $P(4, q) \leqq 3q+1$. These results are to be compared with the following of Wright [Bull. Amer. Math. Soc. **54**, 755–757 (1948); these Rev. **10**, 101]: $P(4, q) \leqq 11$, $P(6, q) \leqq 22$.
D. H. Lehmer (Los Angeles, Calif.).

Gupta, Hansraj. Analogues of some $\mu(n)$ theorems. Math. Student **19**, 19–24 (1951).

Let the Liouville function $\lambda(n)$ be defined by $\lambda(1) = 1$ and $\lambda(n) = (-1)^k$, where k is the number of odd α's in the prime factor decomposition $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$. In his proof of the prime number theorem Selberg [Ann. of Math. (2) **50**, 305–313 (1949); these Rev. **10**, 595] developed a number of formulas involving the Möbius function $\mu(n)$. Following similar lines the author establishes $\lambda(n)$-analogues of these formulas. The main results are: (1) the function $g_k(n) = \sum_{d|n}\lambda(d)\log^k d$ vanishes when k exceeds h; (2) if $k = 0$ then $g_1(n) = \frac{1}{2}\log n$; if $k = 1$, $n = p_1^{\alpha_1}m$, $(m, p_1) = 1$, α_1 is odd, m is a square, then $g_1(n) = -\frac{1}{2}(\alpha_1+1)\log p_1$; (3) for all positive x the sum $\sum_{d|n}\lambda(d)\log^h(x/d)$ is a function of the $(h-k)$th degree in $\log x$ when $k \leqq h$ and vanishes when $k > h$. As an application an asymptotic formula for the sum $\sum_{n \leq x} g_1(n)$ is derived. In conclusion two formulas involving $\lambda(n)$ which resemble the Möbius inversion formula are established. *A. L. Whiteman* (Los Angeles, Calif.).

Balasubrahmanian, N. On the number defined by

$$N_r = \frac{1}{e}\sum_{n=0}^{\infty}\frac{n^r}{n!}.$$

Math. Student **18**, 130–132 (1950).

It is proved that N_r is even if and only if $r \equiv 2 \pmod 3$. As a consequence, $\sum_{r\equiv 2 \pmod 3}\binom{k}{r}$ is even if and only if $k \equiv 1 \pmod 3$. [The first result, and more, is known [G. T. Williams, Amer. Math. Monthly **52**, 323–327 (1945); these Rev. **7**, 47]. The second follows trivially from the explicit evaluation of the sum as $\frac{1}{3}\{2^k + 2(-1)^k\cos\frac{2}{3}\pi(k-1)\}$.]
N. J. Fine (Philadelphia, Pa.).

Manikarnikamma, S. N. Some properties of the series $\sum_{n=0}^{\infty}\frac{(n+a)^r x^n}{n!}$. Math. Student **18**, 132–135 (1950).

The author writes the series as $f_r(x)e^x$, notes that $f_r(x)$ is a polynomial of degree r in x, also in a, and develops various relations among the coefficients. When a and x are integers, x odd, he shows that $f_{3r+1} \equiv a+1$, $f_{3r+2} \equiv a$, $f_{3r} \equiv 1 \pmod 2$. For x even, $f_r \equiv a \pmod 2$ for all r. For $a = 0$, set $f_r(x) = N_r(x)$. Then $x = 1$ yields the first result of the preceding review. Finally,

$$\left(\frac{d}{dx}\right)^r(e^{e^x}) = e^{e^x}N_r(e^x) = \sum_{n=0}^{\infty}\frac{n^r e^{nx}}{n!},$$
$$\left(\frac{d}{dx}\right)^r[e^{(ax+e^x)}] = e^{ax}\sum_{n=0}^{\infty}\frac{(n+a)^r e^{nx}}{n!}.$$

N. J. Fine (Philadelphia, Pa.).

Erdös, Paul, and Shapiro, Harold N. On the changes of sign of a certain error function. Canadian J. Math. **3**, 375–385 (1951).

Let $\varphi(n)$ be Euler's function, $R(x) = \sum_{n=1}^{x}\varphi(n) - 3\pi^{-2}x^2$. The authors prove that neither the upper nor the lower limit of the function $R(x)x^{-1}(\log\log\log\log x)^{-1}$ equals zero, and that $R(x)$ changes sign infinitely often. The proof is based on an "elementary" calculation of the average of the function

$$H(x) = \sum_{n=1}^{x}\varphi(n)n^{-1} - 6\pi^{-2}x,$$

taken over individual residue classes mod $\prod_{p<y}p$. It has been proved by Pillai and Chowla [J. London Math. Soc. 5, 95–101 (1930)] that $R(x)\neq o(\log\log\log x)$.

H. Heilbronn (Bristol).

Subba Rao, M. V. Ramanujan's trigonometrical sum and relative partitions. J. Indian Math. Soc. (N.S.) 15, 57–64 (1951).

For $n<m$, a relative partition of n mod m is defined as a partition of $n+tm$, $t\geqq 0$, into positive parts less than m. The author develops a method for expressing certain weighted restricted relative partition functions in terms of Ramanujan's $C_m(n)=\sum_\rho \exp(2\pi i\rho n/m)$, where ρ runs over a reduced residue system mod m. The restriction on the partitions is that no part is to appear more than ν times; the weights are of the form λ^k, where λ is real and k is the number of parts, or of forms which can be obtained from this one by differentiation with respect to λ. The method is valid, but the results are not (for $\nu>1$), since the author uses the generating function

$$\prod_{l=1}^{r}(1+\lambda x^{a_l})^\nu$$

instead of the correct

$$\prod_{l=1}^{r}(1+\lambda x^{a_l}+\lambda^2x^{2a_l}+\cdots+\lambda^\nu x^{\nu a_l})$$

for the absolute weighted partitions of n into parts $a_1, a_2, \cdots, a_r$, restricted as above. *N. J. Fine.*

Sandham, H. F. Five series of partitions. J. London Math. Soc. 27, 107–115 (1952).

Let $p(n)$ denote the number of partitions of n. The following series are summed explicitly:

$$\sum_1^\infty\frac{p(n)}{\cosh\pi\sqrt{(6(n-1/24))}},$$

$$\sum_1^\infty\left\{\frac{p(n)}{\cosh\pi\sqrt{(\frac{2}{3}(n-1/24))}}-\frac{1}{2n\sqrt{3}}\right\},$$

$$\sum_0^\infty\frac{p(n)}{\cosh\pi\sqrt{(2(n-1/24))}},$$

$$\sum_0^\infty\frac{p(n)}{\cosh\pi\sqrt{(n-1/24)}},$$

$$\sum_0^\infty\frac{p(n)}{\cosh\pi\sqrt{(4(n-1/24))}}.$$

Three other summations, to be proved in later papers, are also noted. The proofs are based on the Euler and Jacobi expansions of $\prod(1-q^n)$ and $\prod(1-q^n)^3$, termwise integration leading to

$$\int_0^1 q^{x+1/24}\prod(1-q^n)\frac{dq}{q}=\frac{\pi\sqrt{2}}{\sqrt{x}}\cdot\frac{\sinh 2\pi\sqrt{(\frac{2}{3}x)}}{\cosh 3\pi\sqrt{(\frac{2}{3}x)}}\quad(x>-\tfrac{1}{24}),$$

$$\int_0^1 q^{x+1/8}\prod(1-q^n)^3\frac{dq}{q}=\frac{2\pi}{\cosh\pi\sqrt{(2x)}}\qquad(x>-\tfrac{1}{8}),$$

application of further identities, and summation of the resulting series by elementary means or by elliptic functions. The method appears to be capable of further application.

N. J. Fine (Philadelphia, Pa.).

Auluck, F. C. On some new types of partitions associated with generalized Ferrers graphs. Proc. Cambridge Philos. Soc. 47, 679–686 (1951).

The partitions considered are of the form

$$n=\sum_{k=1}^{r}a_k+\sum_{j=1}^{s}b_j,$$

$$(1)\qquad\begin{cases}0<a_1\leqq a_2\leqq\cdots\leqq a_r,\quad 0<b_1\leqq b_2\leqq\cdots\leqq b_s,\\ b_s<a_r.\end{cases}$$

The set "b" can be empty. Under Type A, $b_s=a_r-1$, every integer up to a_r is taken at least once in the set "a", and every integer up to b_s is taken at least once in the set "b". Type B has no restrictions other than (1). In Type C, every integer up to a_r is taken at least once in the set "a". The corresponding partition functions are denoted by $P(n)$, $Q(n)$, $R(n)$, and the generating functions are $f(x)$, $g(x)$, $h(x)$, respectively. Then

$$f(x)=\sum_{m=1}^\infty\frac{x^{m^2}}{(1-x)^2(1-x^2)^2\cdots(1-x^{m-1})^2(1-x^m)},$$

$$g(x)=\sum_{m=1}^\infty\frac{x^{m}}{(1-x)^2(1-x^2)^2\cdots(1-x^{m-1})^2(1-x^m)},$$

$$h(x)=\sum_{m=1}^\infty\frac{x^{\frac{1}{2}(m^2+m)}}{(1-x)^2(1-x^2)^2\cdots(1-x^{m-1})^2(1-x^m)}.$$

The following identities are proved:

$$f(x)=\frac{x-x^3+x^6-x^{10}+\cdots}{(1-x)(1-x^2)(1-x^3)\cdots},$$

$$g(x)=\frac{x-x^3+x^6-x^{10}+\cdots}{[(1-x)(1-x^2)(1-x^3)\cdots]^2},$$

$$h(x)=\prod_{k=1}^\infty(1-x^k)^{-1}\sum_{m=0}^\infty\frac{x^{(m+1)(2m+1)}}{(1-x^2)(1-x^4)(1-x^6)\cdots(1-x^{2m})}.$$

Using these, the author expresses $P(n)$, $Q(n)$, $R(n)$ in terms of more familiar partition functions, and obtains estimates

$$P(n)\sim\frac{1}{8n\sqrt{3}}\exp\left\{\pi\left(\frac{2n}{3}\right)^{\frac{1}{2}}\right\},$$

$$Q(n)\sim\frac{1}{8\cdot 3^{3/4}n^{5/4}}\exp\left\{2\pi\left(\frac{n}{3}\right)^{\frac{1}{2}}\right\},$$

$$\pi\sqrt{\tfrac{2}{3}}\leqq n^{-\frac{1}{2}}\log R(n)\leqq\pi\sqrt{\tfrac{4}{3}}.$$

N. J. Fine (Philadelphia, Pa.).

Tanaka, Minoru. An elementary proof of the prime number theorem. Sûgaku (Mathematics) 3, 136–143 (1951). (Japanese)

This is a clear presentation of Selberg's proof of the prime-number theorem [Ann. of Math. (2) 50, 305–313 (1949); these Rev. 10, 595], using, in part, H. N. Shapiro's paper [ibid. 51, 485–497 (1950); these Rev. 11, 419].

S. Ikehara (Tokyo).

Hametner, Herbert. Über die Approximation von indefiniten binären quadratischen Formen. Monatsh. Math. 55, 300–322 (1951).

Let $f(x, y)$ be an indefinite binary quadratic form with real coefficients. The paper is concerned with the determination of numbers M with the property that the inequality $|f(x-x_0, y-y_0)|\leqq M$ is soluble in integers x, y for

any real numbers x_0, y_0. The author's result is somewhat complicated, and depends on the choice of a certain set of integer pairs. The result is applied to the particular cases when $f(x, y)$ is the norm-form of the quadratic fields generated by $\sqrt{47}$, $\sqrt{71}$, $\sqrt{79}$, $\sqrt{53}$, $\sqrt{85}$; and except in the first case smaller values of M are obtained than those previously published by other workers. *H. Davenport* (London).

Jones, Burton W. An extension of Meyer's theorem on indefinite ternary quadratic forms. Canadian J. Math. 4, 120–128 (1952).

Let f, F, $d=|F|$, Ω, Δ be, respectively, an indefinite ternary quadratic form with integral coefficients of g.c.d. 1, its matrix, the determinant of this matrix, the g.c.d. of minors of F of order 2, the integer $\Delta=d/\Omega^2$. A form f^* is called reciprocal form of f, if its matrix F^* is the quotient by Ω of the adjoint matrix adj F of F. The family of forms f having a same pair Ω, Δ is called a genus of ternary quadratic forms. The paper gives certain sufficient conditions, generalizing that, given formerly by Meyer and Dickson, for the (arithmetical) equivalence of all the forms of some given genus, i.e. for the number of classes of this genus being 1.

Meyer and Dickson have proved that such is the case when the g.c.d. of Ω and Δ is 1 or 2, $\Omega\not\equiv 0 \pmod 4$ and $\Delta\not\equiv 0 \pmod 4$. The author, essentially, extends this theorem, under certain supplementary conditions, to the case where the g.d.c. of Ω and Δ has no square factors. The supplementary conditions are the following: If p is a common odd prime divisor of Ω and Δ, there exists an integer q such that: 1) $|q|$ is an odd prime or double of an odd prime; 2) let $f_p(x_1, x_2, x_3)$ be a form equivalent to f, which is congruent (mod p^3) to an expression $a_1x_1^2+p^2a_2x_2^2+pa_3x_3^2$ (it is always possible to find such an f_p); then $-q$ is representable by the form $g^*=[p^{-1}f_p(px_1, x_2, x_3)]^*$ reciprocal to the form $g=p^{-1}f_p(px_1, x_2, x_3)$; 3) every solution of the congruence $x^2-qy^2\equiv 1 \pmod p$ is congruent (mod p) to a solution of the Pell equation $x^2-qy^2=1$. To prove this result, the author proves that if p, but not p^2, divides both Ω and Δ, and if the conditions 1), 2), 3) are satisfied for this p, the genus of f_p has only one class if and only if the genus of $g=p^{-1}f_p(px_1, x_2, x_3)$ has only one. And, if Ω', Δ' is the genus of g, the g.c.d. of Ω', Δ' is equal to that of Ω, Δ divided by p.

In the last part of the paper, it is proved that the condition 2) is satisfied for given f, p, q if it is satisfied by replacing the ring R of rational integers by rings R_l of l-adic integers, where l ranges over all the prime divisors of $2\Delta/p$. Besides, the representability of $-q$ by g^* in R_l is equivalent to satisfying, by q, a certain congruence modulo a certain power of l. The author proves also that in certain particular cases, the condition 3) is satisfied. *M. Krasner.*

Brandt, Heinrich. Über das quadratische Reziprozitätsgesetz. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. 99, no. 1, 17 pp. (1951).

A new formulation of the quadratic reciprocity law is given and discussed in detail. A proof is given based on the second proof of Gauss. *W. H. Mills.*

Brandt, Heinrich. Über das quadratische Reziprozitätsgesetz im rationalen Zahlkörper. Math. Nachr. 6, 125–128 (1951).

Another proof is given of the author's new formulation of the quadratic reciprocity law. *W. H. Mills.*

Krasner, Marc. Une loi de réciprocité (préliminaires). C. R. Acad. Sci. Paris 233, 995–997 (1951).

Krasner, Marc. Une loi de réciprocité. C. R. Acad. Sci. Paris 233, 1409–1411 (1951).

Let k be of finite degree over the rationals, let K/k be finite, normal, and not necessarily abelian, let p_1 and p_2 be finite prime divisors in k which are unramified in K, and let $\Sigma(p_i)$ be their Artin symbols (sets of conjugate elements of the Galois group of K/k). The reciprocity law proved in these papers states that $\Sigma(p_1)=\Sigma(p_2)$ if there exist two irreducible polynomials with coefficients in k which satisfy a certain set of local conditions depending only on K and k and involving only a finite number of prime spots. These conditions are too intricate to reproduce here; but the law is a fairly simple consequence of Artin's (abelian) reciprocity law, Dirichlet's unit theorem, and Krasner's [same C. R. 225, 1113–1115 (1947); 226, 535–537 (1948); these Rev. 9, 326, 408] monodromy law; and this monodromy law is equivalent in content to the following modification of Bauer's theorem: If $K\neq k$ there is a constant C, depending only on k, on degree K/k, and on discriminant of K/k, such that there is a finite prime divisor in k which is unramified in K/k, does not split completely in K/k, and has absolute norm less than C. The proofs in these papers could be simplified by using idèles or Hasse's norm residue symbol. *G. Whaples* (Bloomington, Ind.).

Moriya, Mikao. Rein arithmetischer Beweis über die Unendlichkeit der Primideale 1. Grades aus einem endlichen algebraischen Zahlkörper. J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 164–166 (1950).

The author gives a simple algebraic-number-theoretic proof that in every algebraic number field of finite degree over the field of rational numbers there exist infinitely many prime ideals of degree 1. *E. R. Kolchin.*

Silva, Joseph A. Representation of arithmetic functions in $GF[p^n, x]$ with values in an arbitrary field. Duke Math. J. 19, 31–44 (1952).

Let $GF[p^n, x]$ denote the set of polynomials in an indeterminate x with coefficients in the finite field $GF(p^n)$. By an arithmetic function f is meant a single-valued function defined over $GF[p^n, x]$ with values $f(A)$ in a field $\mathfrak{F}$. An equivalence relation $f\sim g$, meaning $f(A)=g(A)$ for all A such that deg $A<r$, leads to a class of equivalent functions which forms a commutative ring $\mathfrak{R}$ with respect to ordinary addition and one of the three Cauchy multiplications. It has been shown by Carlitz [Duke Math. J. 14, 1121–1137 (1947); these Rev. 9, 337] that if $\mathfrak{F}$ is of characteristic zero and contains the pth roots of unity, then there exists a set of p^{nr} orthogonal (hence linearly independent) functions ϵ_{GH} such that an arbitrary function f in $\mathfrak{R}$ may be represented uniquely by $f=\sum^*\alpha_{GH}\epsilon_{GH}$, where α_{GH} are numbers of $\mathfrak{F}$, and the asterisk indicates a summation over the ϵ_{GH}. The purpose of this paper is to investigate the situation where $\mathfrak{F}$ does not contain the pth roots of unity or is of prime characteristic. In all cases it is found that there exist just p^{nr} linearly independent functions. A main result for the case when $\mathfrak{F}$ has characteristic zero is that an arbitrary function f may be represented uniquely by $f=\sum^*\beta_{GH}\epsilon(1, G, H)$ where the β_{GH} lie in $\mathfrak{F}$, and the asterisk indicates a summation over certain p^{nr} linearly independent functions $\epsilon(1, G, H)$. In the case when the characteristic $q\neq p$ the existence of the functions $\epsilon(1, G, H)$ depends on the non-vanishing of the cyclic determinant $D=|\zeta^{g^{e(j-i)}}|$, $i, j=0, 1, \cdots, f-1$, where ζ is a

primitive pth root of unity, g is a primitive root (mod p), and $p=ef+1$. For characteristic zero, $D\neq 0$. For characteristic $q\neq p$, D may be zero, but it is sometimes possible to choose ζ so that $D\neq 0$. Whether this can always be done is an open question. *A. L. Whiteman* (Los Angeles, Calif.).

Carlitz, L. Some applications of a theorem of Chevalley. Duke Math. J. **18**, 811–819 (1951).

Let $GF(q)$ be the Galois field of q elements, $GF[q, x_1, \cdots, x_s]$ the ring of polynomials in $x_1, \cdots, x_s$ over $GF(q)$, $GF(q, x)$ the field of rational functions in x over $GF(q)$, $GF\{q, x\}$ the field of formal Laurent series

$$\alpha=\sum_{i=-\infty}^{m} c_i x^i, \quad c_i \text{ in } GF(q),$$

where degr $\alpha=m$ if $c_m\neq 0$. The author proves (theorem 2): If $f(x_1, \cdots, x_s)$ in $GF[q, x_1, \cdots, x_s]$ has no constant term and if degr $f\leqq k$, then it is possible to find, for each $w>0$, $U_1, \cdots, U_s$, not all zero, in $GF[q, y_1, \cdots, y_w]$ such that $f(U_1, \cdots, U_s)=0$ provided $s>k^{w+1}$. Moreover, for each q, k, w it is possible to find an f such that the statement is not true for $s=k^{w+1}$.

The author proves various results about approximation. We mention two special cases. If $k\geqq 1$, $m\geqq 1$, α in $GF\{q, x\}$, it is possible to find A and B in $GF[q, x]$ such that

$$\text{degr }(A^k\alpha-B)<-m, \quad \text{degr } A\leqq km.$$

Analogue to Kronecker's Theorem. If $m\geqq 1$, α and θ in $GF\{q, x\}$, θ not in $GF(q, x)$, it is possible to find an A in $GF[q, x]$ such that

$$\text{degr }(A\theta-\alpha)\leqq -\text{degr } A\leqq -m.$$

The proofs are based on a theorem by Chevalley [Abh. Math. Sem. Univ. Hamburg **11**, 73–75 (1935)]. If $F_1, \cdots, F_k$ are polynomials in $GF[q, x_1, \cdots, x_n]$ without constant term and if the sum of their degrees $<n$, it is possible to find $x_1, \cdots, x_n$ in $GF(q)$ not all zero such that $F_1=\cdots=F_k=0$. *H. Heilbronn* (Bristol).

Ankeny, Nesmith C. An improvement of an inequality of Minkowski. Proc. Nat. Acad. Sci. U. S. A. **37**, 711–716 (1951).

The author proves: Let K be an algebraic number field of discriminant Δ and degree n whose Galois group exists and is metabelian in r steps; then $\log|\Delta|\geqq Cnl_r(n)$, where $l_r(n)$ is the logarithm iterated r times. The proof is based on the formula given for the relative discriminant of an abelian extension field by class-field theory. *H. Heilbronn.*

Ankeny, N. C. The least quadratic non residue. Ann. of Math. (2) **55**, 65–72 (1952).

Let p be a prime. Denote by $n(p)$ the least quadratic non residue mod p. Vinogradov proved that

$$n(p)=O(p^{1/2\sqrt{e}}(\log p)^2)$$

and Linnik proved, using the extended Riemann hypothesis, that $n(p)=o(p^{\epsilon})$. The author proves, using the same hypothesis, that $n(p)=O((\log p)^2)$. In view of a remark of Chowla and Turán $n(p)>c\log p$ for infinitely many p; thus the author's estimate is not far from the best possible one.

The author further proves (under the same hypothesis) that the least primitive root $r(p)$ of p satisfies

$$r(p)=O[2^{\nu(p-1)}(\log p)^2(\log[2^{\nu(p-1)}\log p])^2],$$

where $\nu(p-1)$ denotes the number of prime factors of $p-1$. Here Vinogradov proved without any hypothesis that $r(p)=O(p^{\frac{1}{2}+\epsilon})$. The author further states that he can prove (again assuming the extended Riemann hypothesis) that the least dth power non residue (mod p) and the least prime g which is a quadratic residue (mod p), are both $O((\log p)^2)$. *P. Erdös* (Los Angeles, Calif.).

Ankeny, N. C., and Chowla, S. The class number of the cyclotomic field. Canadian J. Math. **3**, 486–494 (1951).

Detailed proof of a result previously announced by the authors [Proc. Nat. Acad. Sci. U. S. A. **35**, 529–532 (1949); these Rev. **11**, 230]. The authors also prove: Under the extended Riemann hypothesis there exists for $\epsilon>0$, $1>s>\frac{1}{2}$, a non-principal character $\chi(n)$ (mod g) such that $|\sum_{n=1}^{\infty}\chi(n)n^{-s}|<1+\epsilon$, provided the prime g is sufficiently large. *H. Heilbronn* (Bristol).

✓***Mahler, K. Farey sections in the fields of Gauss and Eisenstein.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 281–285. Amer. Math. Soc., Providence, R. I., 1952.

A clear review of a theory of Farey sections in $k(i)$ and $k(\rho)$, which meanwhile appeared with full proofs, in a joint paper by Cassels, Ledermann and the author [Philos. Trans. Roy. Soc. London. Ser. A. **243**, 585–626 (1951); these Rev. **13**, 323]. *J. F. Koksma* (Amsterdam).

Basoco, M. A. On a certain arithmetical identity related to the doubly periodic functions of the second and third kinds. Gaz. Mat., Lisboa **12**, no. 50, 11–13 (1951).

For every integer n consider the partitions (a) $n=i^2+2d\delta$, $i\lesseqgtr 0$, $\delta>0$, $d>0$; (b) $n=h^2+\Delta\Delta'$, $h\lesseqgtr 0$, $0<\Delta<\Delta'$, $\Delta\equiv\Delta'$ (mod 2); (c) let $\epsilon(n)=1$ if $n=s^2$, $\epsilon(n)=0$ otherwise; (d) let $\lambda(n)=1$ if $n=r^2+t^2$, $r>0$, $t>0$, $\lambda(n)=0$ otherwise. Then the following identity holds:

$$\begin{aligned}
&4\sum_{n=1}^{\infty}q^n\Big\{\sum_{(a)}\sin 2[(\delta+i)x+(\delta-d+i)y+iz]\Big\}\\
&\quad=4\sum_{n=1}^{\infty}q^n\Big\{\sum_{(b)}\sin[(\Delta+\Delta')x+(\Delta'-\Delta)y+2(\Delta-h)z]\\
&\quad-\sum_{(b)}\sin[-2hx+(\Delta'-\Delta)y+2(\Delta-h)z]\Big\}\\
&\quad+\cot(x+y)\sum_{n=1}^{\infty}q^n\epsilon(n)[\cos 2sz-\cos 2s(x+y+z)]\\
&\quad-\cot y\sum_{n=1}^{\infty}q^n\epsilon(n)[\cos 2s(x+z)-\cos 2s(x+y+z)]\\
&+2\sum_{n=1}^{\infty}q^n\lambda(n)\Big\{\sum_{(d)}[\cos 2rz\sin 2t(x+z)-\cos 2r(x+z)\sin stz]\Big\}.
\end{aligned}$$

Here the summations (a), (b), (d) range over all corresponding partitions of n. The proof uses properties of the theta functions and Cauchy's residue theorem. The right hand side can be transformed so as to contain only terms of the form $\sin(\alpha x+\beta y+\gamma z)$, with α, β, γ integers. Identities of the same kind are indicated for single-valued functions of integral arguments and subject to certain parity conditions. As a particular case the author obtains two formulas due to Uspensky [Bull. Acad. Sci. URSS (6) **1926**, 547–566; Bull. Amer. Math. Soc. **36**, 743–754 (1930)]. *E. Grosswald.*

Spencer, S. M., Jr. Transcendental numbers over certain function fields. Duke Math. J. **19**, 93–105 (1952).

Let F denote a field of arbitrary characteristic and let $x_1, \cdots, x_w$ be indeterminates. The first part of the paper

contains a proof of an analog of a theorem of G. Faber [Math. Ann. 58, 545–557 (1904)]; the analog asserts that certain entire functions $f(t)$ with coefficients in $F(x_1, \cdots, x_w)$ have the property that $f(\alpha)$ is transcendental for all algebraic $\alpha \neq 0$. In the second part of the paper F is assumed to have characteristic p; a number of theorems are proved which extend certain results due to L. I. Wade [same J. 8, 701–720 (1941); 11, 755–758 (1944); these Rev. 3, 263; 6, 144]. As a typical example we quote: the series $\sum_0^\infty G_k^{-q^k}$, where $G_k \epsilon GF[p^n, x_1, \cdots, x_w]$, $G_k | G_{k+1}$, is transcendental for $q \neq p^s$, $q > 1$. *L. Carlitz* (Durham, N. C.).

Myrberg, P. J. Über Primfunktionen auf einer algebraischen Riemannschen Fläche. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 104, 16 pp. (1951).

A prime function on an algebraic Riemann surface R is a function $f(x)$, regular on R, which has a single pole of first order, and such that $f(Kx) = h(x)f(x)$ for every closed circuit K on R, where $h(x)$ is a regular, nowhere vanishing function. The existence of prime functions was proved by Klein and Ritter, who factorized every rational function on R by means of them. In this paper the author sets up prime functions of special type, namely, in which $h(x) = \exp(\lambda u(x) + \eta)$, where u is an "Appell" integral of the first kind:

$$u(Kx) = \alpha u(x) + \beta,$$

and λ, η are constants which are the same for homologous circuits. He represents every rational function on R as a product of such prime functions. In case α is always unity, the Appell integral reduces to an Abelian integral, which the author normalizes so as to make the function $h(x)$ as simple as possible. He then constructs a product of the resulting prime functions which is proportional to the Riemann theta function. *J. Lehner* (Philadelphia, Pa.).

Koksma, J. F. On a certain integral in the theory of uniform distribution. Nederl. Akad. Wetensch., Proc. Ser. A. 54 = Indagationes Math. 13, 285–287 (1951).

Let $\lambda_1 < \lambda_2 < \cdots$ be a sequence of positive integers, and let x be a real number. Denote by $N(\alpha, \beta, x, N)$ the number of integers $n \leqq N$ for which $\lambda_n x$ satisfies $\alpha \leqq \lambda_n x < \beta$ (mod 1). Further put

$$A(N) = \sum_{1 \leqq m, n \leqq N} \frac{(\lambda_m, \lambda_n)}{[\lambda_m, \lambda_n]},$$

where (λ_m, λ_n) denotes the greatest common divisor and $[\lambda_m, \lambda_n]$ denotes the least common multiple of λ_m and λ_n. Define further

$$R(\alpha, \beta, x, N) = N(\alpha, \beta, x, N) - (\beta - \alpha)N.$$

The author proves in a very simple way that

$$(*) \qquad \int_0^1 [R(\alpha, \beta, x, N)]^2 dx < \tfrac{1}{4} A(N).$$

Equation (*) lead the author more than ten years ago to the problem of estimating $A(N)$. This problem was solved by I. S. Gál [Nieuw Arch. Wiskunde (2) 23, 13–38 (1949); these Rev. 10, 355]. *P. Erdös* (Aberdeen).

Cassels, J. W. S. A theorem of Vinogradoff on uniform distribution. Proc. Cambridge Philos. Soc. 46, 642–644 (1950).

Let D denote the discrepancy (mod 1) of the N numbers $a_1, \cdots, a_N$ and E the discrepancy (mod 1) of the N^2 differences $a_n - a_m$ ($n, m = 1, 2, \cdots, N$). Then, by a theorem of Vinogradoff, (1) $D^3 \leqq cE$, where c denotes an absolute constant. The author proves this inequality with $c = 12$ in a short and remarkable way by showing that a certain integral lies between the numbers $N^2D^3/24$ and $N^2E/2$. The reviewer remarks that van der Corput and Pisot [Nederl. Akad. Wetensch., Proc. 42, 476–486 = Indagationes Math. 1, 143–153 (1939)] have replaced (1) by a sharper inequality which contains the estimate $D^{2+\epsilon} \leqq c(\epsilon)E$ for every constant $\epsilon > 0$, $c(\epsilon)$ denoting a constant which depends on ϵ only. *J. F. Koksma* (Amsterdam).

ANALYSIS

Ruderman, H. D. Two new inequalities. Amer. Math. Monthly 59, 29–32 (1952).

Let $a_{ij} \geqq 0$, $i = 1, \cdots, m$, $j = 1, \cdots, n$, and let the a'_{ij}, $j = 1, \cdots, n$, denote the a_{ij}, $j = 1, \cdots, n$, rearranged in decreasing order of magnitude. The author proves the inequalities

$$(1) \quad \sum_j \prod_i a_{ij} \leqq \sum_j \prod_i a'_{ij}, \qquad (2) \quad \prod_j \sum_i a_{ij} \geqq \prod_j \sum_i a'_{ij}.$$

[Of these, (1) is essentially known, since the proof given for its special case $m = 2$ in Hardy, Littlewood and Pólya, Inequalities, Cambridge, 1934, p. 263, applies for any m.] *G. G. Lorentz* (Toronto, Ont.).

Rado, R. An inequality. J. London Math. Soc. 27, 1–6 (1952).

Let $f(x) = \sum x_{\sigma_1}^{\alpha_1} \cdots x_{\sigma_n}^{\alpha_n}$, $g(x) = \sum \prod x_{\sigma_i}^{\beta_i}$, α_i, β_i real, with summation over σ in a group G of permutations of the indices $1, \cdots, n$. Let H be the convex set spanned by the vectors $(\beta\sigma_1, \cdots, \beta\sigma_n)$ for σ in G. Theorem: $f(x) \leqq g(x)$ for all $x_1, \cdots, x_n$ positive if and only if $(\alpha_1, \cdots, \alpha_n)$ lies in H. Further results deal with the multiplicity, under these circumstances, of solutions x of the equation $f(x) = g(x)$. *R. C. Lyndon* (Princeton, N. J.).

Tureckiĭ, A. H. On estimates of approximations of quadrature formulas for functions satisfying a Lipschitz condition. Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 166–171 (1951). (Russian)

The author discusses a special case of a general problem proposed by S. M. Nikol'skiĭ [Uspehi Matem. Nauk (N.S.) 5, no. 2(36), 165–177 (1950); these Rev. 12, 83] concerning the accuracy of mechanical quadratures not for individual functions but for a class of functions. Nikol'skiĭ considered [loc. cit.] classes of functions having a number of derivatives. In this note the author confines his attention to classes Lip α. For example, if $f(x)$ is defined in $[a, b]$ and $|f(x') - f(x'')| \leqq K|x' - x''|^\alpha$, then the maximum error of the trapezoidal formula, for n points, is

$$K \frac{(b-a)^{\alpha+1}}{(\alpha+1)2^\alpha} \frac{1}{(n-1)^\alpha}.$$

Similar estimates are obtained for Simpson's formula, for the case of Tchebychef abscissas, and a few others. *A. Zygmund* (Chicago, Ill.).

Hsu, L. C. Some remarks on a generalized Newton interpolation formula. Math. Student 19, 25–29 (1951).

Démonstration de la formule:

$$\sum_{(m)} f(x_1)f(x_2)\cdots f(x_n)=\left(\sum_{\nu=0}^{k}\Delta^\nu f(-\nu-1)E^\nu\right)^n\binom{x}{m}_{x=m+n-1},$$

où $\Delta f(x)=f(x+1)-f(x)$, $Ef(x)=f(x+1)$, f=polynome de degré k; $\sum_{(m)}$ est à étendre à toutes les solutions entières non négatives de $x_1+\cdots+x_n=m$. Application au calcul de diverses sommes. *J. Favard* (Paris).

Marinescu, G. Application of bilinear functionals to the problem of moments. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 823–827 (1949). (Romanian. Russian and French summaries)

Making use of the representation theorem for bilinear functionals on $L_2\times L_2$ due to Morse and Transue [Proc. Nat. Acad. Sci. U. S. A. 35, 136–143 (1949); these Rev. 10, 612], the author proves the theorem: Given a double sequence of real numbers $\{c_{ik}\}$ and two sequences of functions $\{x_i(t)\}$ and $\{y_k(s)\}$ in L_2. In order that there exist a function $\mu(t, s)$ of bounded variation such that

$$\int_0^1\int_0^1 x_i(t)y_k(s)d\mu(t, s)=c_{ik},$$

it is necessary and sufficient that there exist an $M>0$ such that, for all finite sets of real numbers λ_i and μ_k,

$$|\sum\sum\lambda_i\mu_k c_{ik}|^2\geqq M\int_0^1|\sum\lambda_i x_i(t)|^2dt\cdot\int_0^1|\sum\mu_k y_k(s)|^2ds.$$

E. R. Lorch (New York, N. Y.).

***Mandelbrojt, S. General theorems of closure.** Rice Inst. Pamphlet. Special Issue. The Rice Institute, Houston, Texas, 1951. iii+71 pp.

This monograph gives an exposition (and generalizations) of the results of the author and two collaborators concerning closure of translations and closure of linear combinations of derivatives of a function. The required notions from complex variable theory and from the theory of Banach spaces are explained in the first two chapters. An associated function of a closed set E of points on the real axis in the complex z-plane is defined as follows. Let Δ be the half-plane $y>0$, if E is the entire x-axis, otherwise $\Delta=CE$. Then $M(u)$ is associated with E; if $\varphi(z)$ is regular in Δ, $|\varphi(z)|\leqq M(|z|)/|y|$ in Δ implies φ is identically zero. Chapter 3 deals with closure theorems of which the following is typical: Let $f(x)$ have p derivatives $(0\leqq p\leqq\infty)$, $\int_{-\infty}^{\infty}|f^{(n)}(x)|dx\leqq M_n<\infty$. Let $M(u)=\inf_{0\leqq n\leqq p} M_n u^{-n}$. If $M(u)$ is associated with the zeros of the Fourier transform of the function g of L_1, then $f(x+a)$ belongs to the closed linear manifold of L_1 spanned by linear combinations of the $f^{(n)}(x)$ $(n<p)$ and the translates of g. Since a constant is associated with the empty set, this contains as a particular case $(p=0)$ Wiener's Tauberian theorem. In chapter 4 the results of chapter 3 are applied to the moment problem and to the problem of polynomial approximation on the entire real axis. *W. H. J. Fuchs.*

***Mandelbrojt, S. Quelques théorèmes d'unicité.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 349–355. Amer. Math. Soc., Providence, R. I., 1952.

Summary of work involving the author's "Fundamental Inequality" [Ann. Sci. École Norm. Sup. (3) 63, 351–378 (1946); these Rev. 9, 229] on moment problems, closure of translations, polynomial approximation on the real axis, quasi-analytic functions. [See Bull. Amer. Math. Soc. 54, 239–248 (1948); these Rev. 9, 416; and the pamphlet reviewed above.] *W. H. J. Fuchs* (Ithaca, N. Y.).

Mandelbrojt, S. Théorèmes généraux de fermeture. J. Analyse. Math. 1, 180–208 (1951). (Hebrew summary)

Continuation of the author's previous researches on the closure of sets $\{f^{(n)}(x), g(x+\xi_p)\}_{0\leqq n\leqq N}$. The principal results are contained in his paper reviewed second above. *W. H. J. Fuchs* (Ithaca, N. Y.).

Puig Adam, P. Continued fractions of incomplete differential quotients and their applications. Revista Mat. Hisp.-Amer. (4) 11, 180–190 (1951). (Spanish. French summary)

If $[a, b]$ is an interval, each of f and g a function with range $[a, b]$, and t a number, the statement that w is the continuous continued fraction from a to b determined by (f, g, t) means that w is a number and if ϵ is a positive number there exists a positive number δ such that if $\{x_i\}_{i=0}^{n+1}$ is an ordered subdivision of $[a, b]$ of norm less than δ and $h_i=f(x_i)-f(x_{i-1})$, $k_i=g(x_i)-g(x_{i-1})$, $i=1, 2, \cdots, n+1$, then the continued fraction

$$h_1+\frac{1}{k_1+}\frac{1}{h_2+}\frac{1}{k_2+}\cdots+\frac{1}{h_{n+1}+}\frac{1}{k_{n+1}+}\frac{1}{t}$$

differs from w by less than ϵ. The symbol $\phi_a^b(f, g, t)$ denotes the number w, and $\phi_a^a(f, g, t)$ is defined to be t. (1) If each of f and g is increasing and continuous in $[a, b]$ and t a positive number, then $\phi_a^b(f, g, t)$ exists. (2) If, in addition, f and g are differentiable in $[a, b]$, then the function $\phi_x^b(f, g, t)$ is the solution of the differential equation $z^2g'-f'=z'$ such that $z(b)=t$. The author does not give a valid proof of (2), but the reviewer finds that the statement is true. *H. S. Wall* (Austin, Texas).

Calculus

***Ostrowski, A. Vorlesungen über Differential- und Integralrechnung. Erster Band. Funktionen einer Variablen.** Verlag Birkhäuser, Basel, 1945. xii+373 pp. 43.50 Swiss francs; bound 47.50 Swiss francs.

***Ostrowski, A. Vorlesungen über Differential- und Integralrechnung. Zweiter Band. Differentialrechnung auf dem Gebiete mehrerer Variablen.** Verlag Birkhäuser, Basel, 1951. 482 pp. 63 Swiss francs; bound 67 Swiss francs.

The present volumes constitute the first two parts of the "Vorlesungen". The third is in preparation. They are based on the lectures given by the author on the differential and integral calculus at the University of Basel for some seventeen years. The object is to give a meticulous account of classical infinitesimal analysis and the topics considered are treated in detail and with great care. As the work progresses greater emphasis is put on the theoretical aspects. As the author sees it, the problems encountered in an undertaking of this sort are the acquisition of technical facility for science students and the development of a solid theoretical foundation for the mathematics student. Emphasis is laid on clarity and the author does not hesitate to go into considerable detail and to devote a number of sections to discussion. The tone of the work is set by the first section

which consists of an essay on the nature of mathematics, its objectives, its relation to other disciplines, the special character of mathematical exposition, definition, learning and understanding. Exercises are present in great abundance and vary considerably in difficulty. In general, the examples are of a strictly mathematical nature. Questions of applications are for the most part put aside. The basis for this decision on the part of the author was that a careful discussion of the assumptions used (without which the treatment of such questions would have little pedagogic sense) would expand the size of the work unduly.

The contents of the two volumes are as follows. Volume I. Chapter 1. Introduction: the nature of mathematics, the real number system, consequences of the basic properties of the real numbers, the function concept. Chapter 2. Limits, sequences. Chapter 3. Continuity, trigonometric functions, definite integral. Chapter 4. The derivative, fundamental theorems of the calculus. Chapter 5. The technique of differentiation, inverse functions, the chain rule. Chapter 6. The technique of integration, the logarithm and exponential function. Chapter 7. Applications of the differential calculus to the study of functions, monotoneity, maxima and minima, second derivatives, Taylor formula, expansions of elementary functions. Volume II. Chapter 1. Point sets. Chapter 2. Functions on sets. Chapter 3. Sequences and infinite series. Chapter 4. Complements to the differential calculus, partial differentiation. Chapter 5. Application of the differential calculus to analysis: the implicit function theorem, Jacobians, systems of functions, maxima and minima of functions of several variables. Chapter 6. Numerical computation: interpolation, numerical differentiation and integration, approximate solution of equations. Chapter 7. Arc length, plane curves. Chapter 8. Space curves and surfaces.

M. Heins (Providence, R. I.).

✓*** Severi, Francesco, e Scorza Dragoni, Giuseppe. Lezioni di analisi. Vol. 3. Equazioni differenziali ordinarie e loro sistemi, problemi al contorno relativi, serie trigonometriche, applicazioni geometriche.** Cesare Zuffi, Bologna, 1951. vi+255 pp.

[Volume 1 of this valuable treatise, written by Severi alone, appeared in 1933 with a second edition in 1938 and a third edition in course of publication. Part 1 of vol. 2 appeared in 1942; see these Rev. 10, 238. These two volumes were published by Nicola Zanichelli, Bologna.] The list of contents of the present volume is indicated in the title. The usual local existence and uniqueness theorems for systems of ordinary differential equations are discussed. To this is added a treatment of problems in the large, mainly boundary value problems, based on the fixed point theorems of Brouwer and of Birkhoff and Kellogg. The geometrical applications referred to in the title consist of a brief discussion of the differential geometry of surfaces ending with a two page survey of differential geometry in higher dimensions and projective differential geometry. The mode of presentation is the same as in the earlier volumes, that is, a fairly elementary but rigorous account of the main facts in the body of the text amplified by complements and exercises serving to open vistas into more advanced fields. The complements occupy almost half the book and cover a large number of loosely connected topics in a very skilful manner. There is an abundance of historical material scattered throughout the book.

E. Hille (New Haven, Conn.).

Jacobsthal, E., und Wergeland, H. Über ein Integral aus der Akustik. Norske Vid. Selsk. Skr., Trondheim **1950**, no. 3, 18 pp. (1951).

The author shows that, for $\alpha>0$, $\beta>0$, and $\lambda\geqq 0$,

$$\int_0^\infty \frac{dx}{(\beta x-\lambda \operatorname{ctg} x)^2+\alpha^2}=\frac{\pi}{2\alpha\beta},$$

and

$$\int_0^\infty \frac{dx}{x^2[(\beta x-\operatorname{ctg} x)^2+\alpha^2]}=\frac{\pi}{2\alpha}.$$

G. Piranian (Ann Arbor, Mich.).

Gallego Diaz, J. On the inversion of order in partial elasticities. Gaz. Mat., Lisboa **12**, no. 50, 15–16 (1951). (Spanish)

The partial elasticities $E_x(z)$, $E_y(z)$ of a function $z=z(x, y)$ of independent variables x, y are defined by $z_x'\cdot(x/z)$, $z_y'\cdot(y/z)$, respectively. The author shows that

$$E_x[E_y(z)]=E_y[E_x(z)]$$

provided $z(x, y)=f(xy)$ or $z(x, y)=f(x)\cdot g(y)$.

L. M. Blumenthal (Los Angeles, Calif.).

Theory of Sets, Theory of Functions of Real Variables

*** Sierpiński, Wacław. Algèbre des ensembles.** Monografie Matematyczne. Tom XXIII. Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1951. ii +205 pp.

Ce rapport donne les titres originaux des chapitres, signale les paragraphes non nécessairement impliqués par ces titres ainsi que les résultats, notions ou notations caractéristiques et contient quelques remarques. Chapitre I (§1–6), Algèbre des propositions. Les notations pour les quantificateurs sont $\prod_x P(x)$ et $\sum_x P(x)$. Les problèmes logiques sont mentionnés avec citation des travaux spécialisés. Chapitre II (§7–13), Ensembles, éléments, sous-ensembles. Les questions de la calculabilité d'un nombre, de la "définition" ou "construction" d'un ensemble sont signalées. Chapitre III (§14–22), Opérations élémentaires sur les ensembles. La réunion de deux ensembles, appelée aussi somme, est représentée par $A+B$, l'intersection, appelée aussi produit, par AB, la différence par $A-B$. §19. Parallélisme entre l'algèbre des propositions et l'algèbre des ensembles. Algèbre de Boole. §20. Opération de Stone $A\circ B$ = différence symétrique de A et B. Il est montré que les sous-ensembles d'un ensemble E constituent un groupe abélien vis-à-vis de cette opération. La notion d'anneau booléen n'est pas indiquée. Chapitre IV (§23–30), Fonctions, images d'ensembles, relations. §25. Théorèmes de Banach et de Cantor-Bernstein. §29. La topologie comme chapitre de la théorie générale des ensembles. Chapitre V (§31–39). §35. Théorèmes sur la séparabilité des ensembles. §34. Les opérations de Hausdorff. Ce dernier chapitre contient des notions et propositions qui ne se trouvent pas dans les ouvrages classiques traitant de la théorie des ensembles et qui peuvent être intéressants en théorie de la mesure ou en topologie. Voici deux specimens: Si Φ est un anneau d'ensembles (suivant Hausdorff) et si $E\epsilon\Phi_\sigma$, $H\epsilon\Phi_\delta$ et $H\subset E$, il existe un ensemble P appartenant à la famille $\Phi_\sigma\Phi_\delta$ tel que $H\subset P\subset E$. Si Φ est un corps d'ensembles (suivant Hausdorff), toute paire M, N d'ensembles disjoints de la famille Φ_δ est séparable $\Phi_\sigma\Phi_\delta$, c'est-à-dire qu'il

existe dans $\Phi_1\Phi_2$ deux ensembles disjoints P et R incluant M et N respectivement. L'ouvrage est d'une lecture aisée et agrémenté d'exercices. *C. Pauc* (le Cap).

Kondô, Motokiti. Quelques principes dans la théorie descriptive des ensembles. J. Math. Soc. Japan **3**, 91–98 (1951).

Soit R un ensemble, et $\mathfrak{F}$ une famille d'ensemble dont la réunion est R. Un ensemble $E\subset R$ est dit effectif dénombrablement par rapport à $\mathfrak{F}$ s'il existe une opération analytique [au sens de Kantorovitch et Livenson, Fund. Math. **18**, 214–279 (1932), pp. 224–225] telle que E s'obtienne par application d'une telle opération à une suite dénombrable d'ensembles de $\mathfrak{F}$. Une proposition P est dite effective dénombrablement par rapport à $\mathfrak{F}$ si P est équivalente à une proposition portant sur un nombre dénombrable d'ensembles effectifs dénombrablement par rapport à $\mathfrak{F}$. L'auteur donne deux principes, dits principes sur l'abstraction, permettant d'obtenir des démonstrations, qu'il appelle analytiques, de telles propositions. Applications aux ensembles projectifs. *A. Appert* (Angers).

Andreoli, Giulio. Sulla geometria di certi insiemi perfetti piani, e su certi numeri non Archimedei. Ricerca, Napoli **1**, no. 4, 17–23 (1950); **2**, no. 1, 9–21 (1951).

In a circle C_0 of radius 1 let A_k ($k=0, 1, \cdots, n-1$) be the vertices of an inscribed regular polygon of n sides. At each A_k a circle C_{1k} ($k=0, 1, \cdots, n-1$) of a sufficiently small radius ρ_1, touching C_0 from inside, is drawn such that the circles C_{1k} are disjoint. On each of the C_{1k}, starting with A_k, the vertices of a regular n-polygon are marked and at each of these vertices a circle of a sufficiently small radius ρ_2, touching C_{1k} from inside, is drawn. This process is continued indefinitely. The set of all the vertices (of all the successive polygons) and of their points of accumulation form a perfect set H_n. By an analogous process (using homotheties with reciprocal ratios) the construction is extended outside C_0, altogether generating a set H_n^* in the plane. The author studies this set H_n^*, the groups connected with H_n^*, and the n-adic numbers (forming a non-Archimedean field) related to H_n^*. *A. Rosenthal* (Lafayette, Ind.).

✱Kurepa, Georges. Problématique des ensembles partiellement ordonnés. Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 7–22. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

Expository paper. *E. Hewitt* (Seattle, Wash.).

Arens, Richard. Ordered sequence spaces. Portugaliae Math. **10**, 25–28 (1951).

Let I_0 be the closed unit interval of real numbers in their natural order, and let I_n be the lexicographically ordered set of all ordinary sequences of elements of I_{n-1} ($n=1, 2, 3, \cdots$). The present note is devoted to showing that for $m\neq n$ ($m, n=0, 1, 2, \cdots$) I_m and I_n are not similar. This, however, has already been proved by Hausdorff [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **58**, 123–143 (1906), p. 143]. *F. Bagemihl* (Rochester, N. Y.).

Ghika, Al. Uniform inductive ordered sets. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 119–124 (1950). (Romanian. Russian and French summaries)

A non-void partially ordered set E is uniform inverse inductive if every non-void subset of E contains at least one minimal element. Main theorem: if $P\subset E$, if P contains all minimal elements of E, and if P contains x whenever it contains all $s<x$, then $P=E$. *E. Hewitt.*

Aigner, Alexander. Eine kombinatorische Systematik der Punktmengen. Elemente der Math. **7**, 11–14 (1952).

Let X be a subset of the real line. Consider the set X, the set of its limit points, and the set of limit points of its complement. Classify a real point relative to X according to which of these 3 sets it does or does not belong. Of the 2^3 combinatorially possible point types 6 actually occur. Classify a set x according to which of these 6 point types relative to it do or do not occur. Of the 2^6 combinatorially possible set types 27 actually occur. *W. Gustin.*

Aigner, Alexander. Der multiplikative Aufbau beliebiger unendlicher Ordnungszahlen. Monatsh. Math. **55**, 297–299 (1951).

The theorem obtained in this paper is a generalization, to arbitrary ordinals, of the result of an earlier paper [Monatsh. Math. **55**, 157–160 (1951); these Rev. **13**, 120], but is not new; cf. Cantor [Gesammelte Abhandlungen, Springer, Berlin, 1932, p. 343] and Jacobsthal [Math. Ann. **66**, 145–194 (1909), p. 184]. *F. Bagemihl.*

Bagemihl, Frederick. A theorem on intersections of prescribed cardinality. Ann. of Math. (2) **55**, 34–37 (1952).

The author proves a general theorem in set theory, the most important special case of which can be stated as follows: With every line l in the plane we associate a cardinal number a_l satisfying $2\leqq a_l\leqq\aleph_0$. Then there exists in the plane a set of points which intersects every line l in precisely a_l points. *P. Erdös* (Los Angeles, Calif.).

Halfar, Edwin. A note on point-set operators. Portugaliae Math. **10**, 103–104 (1951).

The author continues the study of the reduction of operator combinations—a subject which has been investigated recently by Chittenden, Sanders and Stopher. He concerns himself with a special case of operators of the form $t(u\cdot v)$, where $u\cdot v$ is defined by $(u\cdot v)A=uA\cap vA$. for every set A. If u and v are any two operators, $d(u\cdot v)$ and $cd(u\cdot v)$ are incapable of further simplification. However, under certain restrictions further applications of d and c give combinations that can be replaced by combinations of operators that do not involve $u\cdot v$. As one example, if u is an operator satisfying $duA\,CuA$ then for any operator v, $dcd(u\cdot v)A=dcduA\cup dcdvA$. An example is given to prove that the condition is necessary. *J. R. Kline* (Philadelphia, Pa.).

Sodnomov, B. S. On arithmetic sums of sets. Doklady Akad. Nauk SSSR (N.S.) **80**, 173–175 (1951). (Russian)

Let E_1 and E_2 be subsets of the real line R. The arithmetic sum E_1+E_2 is defined as the set of all $x+y$, for $x\epsilon E_1$ and $y\epsilon E_2$. Let T denote the smallest family of subsets of R containing all Borel sets and closed under the formation of arithmetic sums. The author states and sketches proofs for the following. (1) For every ordinal number $\alpha<\Omega$, there exists a projective set of class CA_α and a perfect set whose arithmetic sum is of class $A_{\alpha+1}$. (2) There exist a set of type G_δ and a perfect set whose arithmetic sum is an analytic set which is not measurable Borel. (3) If every set in T is Lebesgue measurable, then every projective set is Lebesgue measurable. *E. Hewitt* (Seattle, Wash.).

Dowker, Yael Naim. Finite and σ-finite invariant measures. Ann. of Math. (2) 54, 595–608 (1951).

In this paper it is shown that a necessary and sufficient condition that a transformation admit a finite (or σ-finite) invariant measure equivalent to a given measure is that the conclusion of the Birkhoff (or Hopf) ergodic theorem hold for certain classes of functions. Let T be a one-to-one measurable transformation of a totally finite measure space (X, F, m) such that $m(T(A))=0$ if and only if $m(A)=0$. Write $\mu\sim m$ if μ is a measure on (X, F) having the same nullsets as m. There exists a σ-finite T-invariant measure $\mu\sim m$ if and only if (1) there exists a positive measurable function g such that for every measurable function f, $0\leqq f(x)\leqq g(x)$, the ratio $\sum_0^n f(T^i x)/\sum_0^n g(T^i x)$ converges a.e. as $n\to\infty$, or (2) there exists a covering sequence of sets A with the property that for every $B\epsilon F$, $B\subset A$, the $\lim\left(\sum_0^n \chi_B(T^i x)/\sum_0^n \chi_A(T^i x)\right)$ exists a.e., or (3) there exists a positive measurable function f such that $\sum_0^n f(T^i x)/\sum_0^n \omega_i(x)$ has a positive finite limit superior (or inferior) a.e., where $\omega_i=d(mT^i)/dm$. Corresponding necessary and sufficient conditions for the existence of a finite invariant measure $\mu\sim m$ are that (1) hold with $g(x)=1$, or that (2) hold with $A=X$. In this case one need only define $\mu(A)=\lim n^{-1}\sum_0^n m(T^{-i}A)$, and apply the Vitali-Hahn-Saks theorem. *J. C. Oxtoby.*

Yosida, Kôsaku, and Hewitt, Edwin. Finitely additive measures. Trans. Amer. Math. Soc. 72, 46–66 (1952).

In §1 the authors consider a set T, a family $\mathfrak{M}$ of subsets closed under finite unions and complements, and the set Φ of all real measures $\phi(E)$, finitely additive and bounded on $\mathfrak{M}$, which form a vector lattice under the natural partial ordering $\phi\geqq 0$. $\phi\epsilon\Phi$ is said to be "purely finitely additive" if for any countably additive measure $\psi\geqq 0$, $\psi\wedge(\phi\vee 0)=\psi\wedge((-\phi)\vee 0)=0$ ($\vee$, $\wedge$ lattice operations). These measures also form a vector lattice and any $\phi\epsilon\Phi$ can be uniquely written as the sum of a countably additive measure ϕ_c and a purely finitely additive measure ϕ_p. In §2 a subfamily $\mathfrak{N}\subset\mathfrak{M}$ of "null sets" is chosen, closed under countable unions and such that $N\epsilon\mathfrak{N}$, $A\subset N$ imply $A\epsilon\mathfrak{N}$; $\mathfrak{L}_\infty$ denotes the Banach space of all $\mathfrak{M}$-measurable functions $x(t)$, $t\epsilon T$, with norm $\|x\|_\infty=\text{ess sup}|x(t)|$, where the ess sup is defined in the usual way using as null sets the sets $N\epsilon\mathfrak{N}$. The elements of the conjugate space $\mathfrak{L}_\infty^*$ are the $\phi\epsilon\Phi$ such that $\phi(N)=0$ on $\mathfrak{N}$, and this proves the existence of many measures ϕ_p. In §3 it is shown that if T is the real line and $\mathfrak{M}$, $\mathfrak{N}$ the ordinary Lebesgue families, then there exist measures ϕ, $\psi\epsilon\mathfrak{L}_\infty^*$ ($\phi\neq 0$) such that $\phi(x)=0$ for every bounded continuous function $x(t)$, and $\psi(x_t)=x(t+a)$ (a=any given number, $x_t(u)=x(t+u)$), and also that Fubini's theorem is false for measures ϕ_p. In §4 $\mathfrak{L}_\infty$ is represented as the algebra of all continuous functions on a compact Hausdorff space Ω, and $\mathfrak{L}_\infty^*$ as the set of all countably additive, regular, Borel measures $\bar{\phi}$ on Ω, in such a way that a measure ϕ_c is countably additive if and only if the corresponding $\bar{\phi}_c$ vanishes for every nowhere-dense closed G_δ on Ω. The characterization of the $\bar{\phi}_p$ which correspond to the purely finitely additive measures ϕ_p is a little more complicated.

M. Cotlar (Chicago, Ill.).

Kawada, Yukiyosi. Correction to my paper on equivalence of measures on an infinite product space. Math. Japonicae 2, 102 (1951).

Correction of theorem 7 in Math. Japonicae 1, 170–177 (1949); these Rev. 11, 89.

✓★**Burkill, J. C. The Lebesgue integral.** Cambridge Tracts in Mathematics and Mathematical Physics, no. 40. Cambridge, at the University Press, 1951. viii+87 pp. 12s. 6d.

The author states: "My aim is to give an account of the theory of integration due to Lebesgue in a form which may appeal to those who have no wish to plumb the depths of the theory of real functions. There is no novelty of treatment in this tract; the presentation is essentially that of Lebesgue himself. The groundwork in analysis and calculus with which the reader is assumed to be acquainted is, roughly, what is in Hardy's A course of pure mathematics [Cambridge Univ. Press, 1908]." The author's treatment of measure and integration is similar in terminology, notation, approach, and scope to that in Titchmarsh's The theory of functions [Oxford, 1932]. The book contains the definition of the Lebesgue integral as the measure of the ordinate set; a brief treatment of multiple integrals, including Fubini's theorem; a chapter on the Lebesgue-Stieltjes integral for functions defined on the line; exercises with hints and solutions; and no bibliography and few references to the literature. *G. B. Price* (Lawrence, Kan.).

Tsuchikura, Tamotsu. Some remarks on the Riemann sums. Tôhoku Math. J. (2) 3, 197–202 (1951).

A function $f(x)$ of period 1 and class L is said to have R if $n^{-1}\sum_{k=1}^n f(x+k/n)\to\int_0^1 f(u)du$ for almost all x. If $f\sim\sum c_n e^{inx}$, it is known that $\sum|c_n{}^2 n^\epsilon|<\infty$ implies R. Inter alia, the author proves R for almost all $f_t(x)\sim\sum\pm c_n e^{inx}$ for which $\sum'|c_n{}^2|\log|n|<\infty$ and all f for which $|c_n|$ is monotone and $\sum'|c_n{}^2|\log|n|<\infty$. *H. D. Ursell* (Leeds).

Parthasarathy, M., and Rajagopal, C. T. A theorem on the Riemann-Liouville integral. Math. Z. 55, 84–91 (1951).

The authors prove, as a companion to a theorem of M. Riesz [Acta Litt. Sci. Szeged. 1, 114–126 (1923)] the following result with a one-sided hypothesis. Let $\phi(x)$ be of bounded variation in every finite interval,

$$\Gamma(\alpha)\Phi_\alpha(x)=\int_0^x (x-t)^{\alpha-1}\phi(t)dt, \qquad \alpha>0,$$

$1\leqq\gamma<r$, min $(k, 1+r)\geqq r-1$. Then the hypotheses

$$\phi(x)=O_L(x^l) \quad\text{and}\quad \Phi_r(x)\sim Kx^k$$

as $x\to\infty$ together imply $\Phi_\gamma(x)=o\{x^{[l(r-\gamma)+k\gamma]/r}\}$ if $k<l+r$,

$$\Phi_\gamma(x)\sim K\frac{\Gamma(k+1)}{\Gamma(k+r+\gamma+1)}x^{k-r+\gamma}$$

if $k\geqq l+r$. The second case would be trivial with a two-sided hypothesis. The case of positive integral r implies the following theorem about a function and its derivatives. If $f(x)$ is the rth integral of $f^{(r)}(x)$, $f^{(r)}(x)=O_L(x^l)$ and $f(x)\sim Kx^k$, then $f^{(r-1)}(x)=o(x^{[l(r-1)+k]/r})$ if $k<1+r$,

$$f^{(r-1)}(x)\sim Kk(k-1)\cdots(k-r+2)x^{k-r+1}$$

if $k\geqq l+r$. *R. P. Boas, Jr.* (Evanston, Ill.).

Viola, Tullio. Sulla formula d'integrazione per parti nell'integrazione doppia secondo Stieltjes. Giorn. Mat. Battaglini (4) 4(80), 142–158 (1951).

It is shown that a necessary and sufficient condition for the validity of the integration by parts formula for Lebesgue-Stieltjes integrals, viz:

$$\int_a^b fdg=f(b+0)g(b+0)-f(a-0)g(a-0)-\int_a^b gdf$$

for f and g of bounded variation in x, is that

$$\sum_x[f(x+0)+f(x-0)-2f(x)]\cdot[g(x+0)-g(x-0)] = -\sum_x[g(x+0)+g(x-0)-2g(x)]\cdot[f(x+0)-f(x-0)],$$

the summation being extended over all discontinuities of f and g. An equivalent result is due to B. Finetti and M. Jacob [Giorn. Ist. Ital. Attuari 6, 303–319 (1935)]. Integration by parts theorems are derived for functions of bounded variation according to the definition of Vitali ($\sum|\Delta_x\Delta_y f|$ bounded for all subdivisions of a rectangle, with $f(x,y)$ of bounded variation in x for each y and in y for each x). These formulas involve the functions $f(x+0,y+0)$, $f(x-0,y+0)$, $f(x+0,y-0)$, $f(x-0,y-0)$ and similarly for g. The method of derivation adapts the procedure of Saks [Theory of the Integral, Warsaw-Lvov, 1937, p. 102] using a Fubini theorem. Necessary and sufficient conditions for the validity of an integration by parts theorem for $f(x,y)$ and $g(x,y)$ then follow easily, the conditions involving the discontinuities of f and g. The procedure can be extended to n variables. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Calderón, Alberto P. On the differentiability of absolutely continuous functions. Rivista Mat. Univ. Parma 2, 203–213 (1951).

It was shown by L. Cesari [Ann. Scuola Norm. Super. Pisa (2) 10, 91–101 (1941); these Rev. 3, 230] that if, on the square $S[0,1;0,1]$, $f(x,y)$ is continuous and absolutely continuous in Tonelli's sense (ACT), and if its partial derivatives are in L^p $(p>2)$, then $f(x,y)$ has a total differential, in the sense of Stolz, almost everywhere in S, but that in the case $p=2$ there may be no point in S at which $f(x,y)$ has a total differential. The author generalizes these results as follows. Let $f(x_1,\cdots,x_n)$ be continuous and ACT on the hypercube K: $0\leqq x_i\leqq 1$, $i=1,\cdots,n$; let $\phi(t)$ be a non-negative convex increasing function, for $0\leqq t<\infty$, such that $\phi(0)=0$, and let grad f belong to L_ϕ on K (that is, let $\int_K\phi[|\text{grad } f|/\lambda]dv<\infty$ for some constant λ depending on f, where dv is the element of hypervolume). Then (i) if $\int_1^\infty[t/\phi(t)]^{1/(n-1)}dt<\infty$ the total differential of f must exist almost everywhere in K, (ii) if this integral is divergent the total differential may be non-existent almost everywhere in K. *H. P. Mulholland* (Birmingham).

Cafiero, Federico. Sugli insiemi compatti di funzioni misurabili negli spazi astratti. Rend. Sem. Mat. Univ. Padova 20, 48–58 (1951).

A theorem of Fréchet states that a family Φ of Lebesgue measurable functions on a set E is compact with respect to convergence in measure (μ) if and only if given $\epsilon>0$ there is a partition of the set E, $E=\sum_1^r E_i$, a number A and a family of sets I_f, $f\epsilon\Phi$, such that $\mu(I_f)<\epsilon$, $|f|<A$ in $E-I_f$, and the oscillation of each function f on each set E_i-I_f is less than ϵ. Here μ is the ordinary Lebesgue measure. The author shows that Fréchet's theorem holds when μ is a non-negative monotone and countably sub-additive function of sets on a general additive family of sets, and derives criteria of compactness for quasi-uniform convergence in capacity and other types of convergence of functions of two variables. *M. Cotlar* (Chicago, Ill.).

Rothberger, Fritz. A remark on the existence of a denumerable base for a family of functions. Canadian J. Math. 4, 117–119 (1952).

A family F of real-valued functions on domain X is said to have a denumerable base provided there exists a sequence $\{f_n\}$ of real-valued functions on X (not necessarily belonging to F) such that each $f\epsilon F$ is the pointwise limit of a subsequence of $\{f_n\}$. The author has previously proved [Ann. of Math. (2) 45, 397–406 (1944); these Rev. 6, 120] that if the continuum hypothesis is true then so is the proposition P which asserts that if F and X each have the power c of the continuum, then F has a denumerable base. In connection with the question whether P can be established without the continuum hypothesis, he now proves a theorem implying that P is false if any of the following three statements holds:

(1) $2^{\aleph_1}=\aleph_{\omega_2}$ and $\aleph_2\leqq c<\aleph_{\omega_1}$;

(2) $2^{\aleph_2}=\aleph_{\omega_2}$ and $\aleph_2\leqq c<\aleph_{\omega_2}$;

(3) $2^{\aleph_n}=\aleph_{\omega_{n+1}}$ $(n=0,1,2,\cdots)$.

T. A. Botts (Charlottesville, Va.).

Theory of Functions of Complex Variables

Epstein, Bernard, and Lehner, Joseph. On Ritt's representation of analytic functions as infinite products. J. London Math. Soc. 27, 30–37 (1952).

J. F. Ritt [Math. Z. 32, 1–3 (1930)] proved that if $f(z)$ is analytic at the origin, and if $f(0)=1$, then $f(z)$ has a representation $\prod_{n=1}^\infty(1+a_nz^n)$, valid in a circular neighborhood N of the origin; Ritt also proved that if $f'(z)/f(z)=\sum_{n=1}^\infty c_nz^{n-1}$, and if $r=\sup|c_n|^{1/n}$, then the radius of N is $\geqq 1/6r$. The authors show that the radius of N is $\geqq 1/r$; this is the best possible result. The authors' proof is modeled on Ritt's manner of studying the recurrence relations which express the a_n in terms of the c_n, but is much more precise and detailed; unlike Ritt's proof, it separates the inductive discussions of the a_n into several categories according to number-theoretic properties of n. *W. Strodt.*

Kaufmann, I. On analytic, uniform, everywhere continuous functions with a perfect and totally discontinuous set of singularities. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 569–571 (1949). (Romanian. Russian and French summaries)

Let M be the set of singularities of an analytic uniform and everywhere continuous function $f(z)$; assume that M is perfect and totally discontinuous (for proof that such functions $f(z)$ exist see Pompeiu [C. R. Acad. Sci. Paris 139, 914–915 (1904)] and A. Denjoy [ibid. 148, 1154–1156 (1909)]. Denote by S and call portion, any closed subset of M, such that also $M-S$ is closed. The author announces the following results: (1) $f(z)$ does not take the same value in all points of a portion S of M (this is equivalent to known results of S. Stoïlow [Bull. Math. Soc. Roumaine Sci. 38, 117–120 (1936)] and W. Fédoroff [C. R. Soc. Sci. Varsovie 24, 92–108 (1932)]; (2) if S is a portion of M, there exist points $\zeta_1\epsilon S$, $\zeta_2\epsilon M-S$ such that $f(\zeta_1)=f(\zeta_2)$; (3) if $G\subset M$ such that $f(G)$ is the boundary of $f(M)$ and $f(M)$ does not belong to more than one sheet of the Riemann surface, then G does not contain portions. In the proofs, use is made of a theorem of W. Wolibner [ibid. 25, 56–62 (1933)] and of several lemmas of the author. *E. Grosswald.*

Lech, Christer. On the coefficients in the power series expansion of a rational function with an application on analytic continuation. Ark. Mat. 1, 341–346 (1951).

The following theorem is established: Let $r(x)$ be a rational function, which in the neighborhood of $x=0$ is repre-

sented by a power series with rational coefficients, whose reduced forms are α_n/β_n,

$$r(x)=\frac{\alpha_0}{\beta_0}+\frac{\alpha_1}{\beta_1}x+\cdots+\frac{\alpha_n}{\beta_n}x^n+\cdots,$$

α_n, β_n integers, $(\alpha_n,\beta_n)=1$ $(n=1,2,3,\cdots)$ $\beta_n=1$, when $\alpha_n=0$. Then, either the sequence $|\beta_n|$ $(n=1,2,3,\cdots)$ is bounded or $\lim\sup_{n\to\infty}|\beta_n|^{1/n}>1$. Using this and a result of F. Carlson, it is shown that $f(z)=\sum a_nz^n$ cannot be continued across the unit circle, if (a) $f(z)$ is regular and one-valued in this circle, except perhaps at a finite number of points, other than 0, and (b) $a_n=\alpha_n/\beta_n$, where α_n, β_n are coprime integers, with $\lim\sup|\beta_n|=\infty$, $\lim\beta_n/n=0$. This generalizes known theorems of similar type where the a_n's are rational integers. [The function $\log(1-z)$ shows that the condition $\beta_n/n\to0$ cannot be relaxed. The reviewer thinks that the proof is valid for α_n, β_n integers of any imaginary quadratic field, and also for infinitely many isolated singularities in $|z|<1$, as in Pólya's generalization [Proc. London Math. Soc. (2) 21, 22–38 (1922)] of Carlson's theorem. It would be of interest to know whether the whole Pólya theorem, with unit circle replaced by an arbitrary region having image-radius 1, can be also established for $a_n=\alpha_n/\beta_n$.] *R. N. Redheffer* (Los Angeles, Calif.).

Walsh, J. L. Note on approximation by bounded analytic functions. Proc. Nat. Acad. Sci. U. S. A. 37, 821–826 (1951).

Soit C une courbe de Jordan, analytique, fermée, simple, intérieure à un domaine D du plan des z; considérons une suite de fonctions analytique, telles que

$$|f_n(z)|\leqq AR^n \quad (z\epsilon D,\ A \text{ constante, } R>1,\ n=1,2,\cdots).$$

S'il existe une fonction $f(z)$ telle que

$$|f-f_n|\leqq B/n^{p+\alpha} \quad (z\epsilon C,\ B \text{ constante, } p \text{ entier } \geqq 0,\ 0<\alpha\leqq 1),$$

alors, sur C, la dérivée $f^{(p)}(z)$ existe et satisfait à une condition de Lipschitz d'ordre α, si $\alpha<1$, ou bien appartient à la classe Λ^* par rapport à l'arc de C $(\alpha=1)$.

Soit $f(z)$ régulière dans C et continue sur C, $F_M(z)$ une fonction régulière dans D et de module au plus égal à M, posons

$$\mu_M(f)=\min_{F_M}\{\max_{z\epsilon C}|f-F_M|\}.$$

Alors, pour que $f^{(p)}(z)$ satisfasse à une condition de Lipschitz d'ordre α $(\alpha<1)$, ou appartienne à Λ^* $(\alpha=1)$, il est nécessaire et suffisant que $\log\mu_M+(p+\alpha)\log\log M$ reste borné lorsque M augmente indéfiniment. Extension au cas où C est un arc analytique simple (les résultats sont vrais pour l'intérieur de l'arc; application aux fonctions périodiques d'une variable réelle. *J. Favard* (Paris).

Lohin, I. F. On completeness of a system of functions of the form $\{F(\lambda_nz)\}$. Doklady Akad. Nauk SSSR (N.S.) 81, 141–144 (1951). (Russian)

Let $0<\lambda_1<\lambda_2<\cdots$, $\lim\sup_{R\to\infty}(1/\log R)\sum_{\lambda_j<R}\lambda_j^{-1}=\alpha>0$. Let $F(z)=\sum_{n=0}^\infty a_nz^n/n!$ be an entire function of order one, mean type, $a_k\neq0$ $(k=0,1,2,\cdots)$, and suppose that $f(z)=\sum_{n=0}^\infty a_nz^{-n-1}$ is regular in the z-plane with a straight line cut from -1 to 1. Then $\{F(\lambda_nz)\}$ is L^p-complete $(1\leqq p\leqq\infty)$ in every closed bounded domain inside $|\text{Im}(z)|<\alpha\pi$. The proof uses the standard method of investigating the function $g(z)=\int_C\gamma(s)F(sz)ds$ (s arc length of curve C lying in $|\text{Im}(s)|<\alpha\pi$). *W. H. J. Fuchs* (Ithaca, N. Y.).

Gahov, F. D. On singular cases of Riemann's boundary problem. Doklady Akad. Nauk SSSR (N.S.) 80, 705–708 (1951). (Russian)

This paper continues the work of some earlier papers of the author on the same problem [same Doklady 67, 601–604 (1949); these Rev. 11, 169; Izvestiya Akad. Nauk SSSR. Ser. Mat. 14, 549–568 (1950); these Rev. 12, 402]. It extends the solution to the case where the elements of the matrix A become infinite of some integral order and where the determinant of A may vanish, both at a finite number of points on L. By a transformation of A to a canonical form similar to that used in the preceding papers the author finds that the number of linearly independent solutions of the homogeneous problem is not affected by the existence of zeros of $|A|$ but diminished according to the number of "poles" of the matrix elements, and that the integrability conditions for the nonhomogeneous problem are correspondingly changed. *M. Golomb* (Lafayette, Ind.).

Hayman, W. K. A characterization of the maximum modulus of functions regular at the origin. J. Analyse Math. 1, 135–154 (1951). (Hebrew summary)

Let $f(z)$ be analytic and regular near $z=0$ and set $M(r)=\max_{|z|=r}|f(z)|$. The author solves the problem of the local characterization of $M(r)$ near $r=0$. He begins by showing that the points where $|f(z)|$ attains its maximum lie among the points where $zf'(z)/f(z)$ is real. The set of points near the origin where a regular function $g(z)=a_kz^k+a_{k+1}z^{k+1}+\cdots$ with $k>0$ and $a_k\neq0$ is real consists of $2k$ regular arcs emanating from $z=0$. He then proves the following theorem characterizing $M(r)$: If $f(z)=1+a_kz^k+\cdots$, $a_k\neq0$, is regular at $z=0$, then

$$M(r)=1+|a_k|r^k+\cdots$$

is regular near $r=0$, and continuing $M(r)$ into the complex plane, there is an $\epsilon>0$ such that $|M(re^{i\theta})|\leqq M(r)$ for $0\leqq r\leqq\epsilon$, $0\leqq\theta\leqq2\pi$. Also $|f(z)|=M(r)$ at most on k regular arcs making angles of $2p\pi/k$ with each other at $z=0$. Next an equivalent criterion characterizing $M(r)$ is given which depends only on the power series expansion of $M(r)$ near $r=0$. In the second part of the paper, the author characterizes all functions with a given maximum modulus. If $M(r)=1+a_kr^k+\cdots$, $a_k\neq0$, then for $k=1$ or 2 there exists a unique function $f(z)$ with $\max_{|z|=r}|f(z)|=M(r)$ and attaining its maximum modulus for small r on an assigned regular arc through the origin. That this result is not true for $k\geqq3$ is also demonstrated. *G. Springer.*

Hayman, W. K. Some applications of the transfinite diameter to the theory of functions. J. Analyse Math. 1, 155–179 (1951). (Hebrew summary)

The author in this paper uses Fekete's transfinite diameter [Math. Z. 17, 228–249 (1923)] to prove various analogues of the "distortion" theorems for functions regular or meromorphic in $|z|<1$ and also for certain classes of "weakly" p-valent functions. Some of these and similar results were obtained by the author in a previous paper [Symmetrization in the theory of functions, Stanford Univ., Calif., 1950; these Rev. 12, 401] using the method of symmetrization. The basis for this investigation is the theorem: Suppose $w=f(z)=z^{-1}+a_0+a_1z+\cdots$ is meromorphic in $|z|<1$ and has a simple pole with residue 1 at the origin. Let D_f be the domain of all values w taken by $f(z)$ in $|z|<1$ and let E_f be the complement of D_f in the closed plane. Then $d(E_f)$, the transfinite diameter of E_f, satisfies $d(E_f)\leqq1$, equality holding if and only if $f(s)$ is univalent. This leads to the

theorem: Let e be a bounded closed set of real numbers x whose Lebesgue measure is at least 4. For each x in e, let $C(x)$ be a closed set of points in the w-plane such that if w_1, w_2 are any points on $C(x_1)$ and $C(x_2)$ resp., we have $|w_1-w_2|\geqq|x_1-x_2|$. Then D_f contains at least one of the sets $C(x)$, except possibly when e and E_f are intervals of length 4 and $f(z)=z^{-1}+a_0+ze^{i\lambda}$, a_0 arbitrary and λ real. This theorem is then applied to yield precise bounds for $|a_1|$, $|f(z)|$ and $|f'(z)|$ for functions $f(z)=a_0+a_1z+\cdots$, regular in $|z|<1$ [cf. report cited above]. A function $f(z)$ defined in a simply connected domain Δ is called weakly p-valent if for every $r>0$, the equation $f(z)=w$ either (i) has exactly p roots for every w on the circle $|w|=r$, or (ii) has less than p roots in Δ for some w on $|w|=r$. For weakly p-valent functions $f(z)=a_0+a_1z+\cdots$ in $|z|<1$ with $f(z)\neq 0$ or with a single p-tuple zero at $z=0$, the author obtains the precise bounds for $|f(z)|$ and $|f'(z)|$ given for mean p-valent functions in the report cited above.

G. Springer (Evanston, Ill.).

Hayman, W. K. The maximum modulus and valency of functions meromorphic in the unit circle. Acta Math. 86, 89–191, 193–257 (1951).

Let $w=f(z)$ be meromorphic in $|z|<1$, and in $|z|<\rho$, $0<\rho<1$, have valency at most $p(\rho)\geqq 1$ on a closed set E containing $w=0$, ∞ and at least one other finite value. The author obtains bounds for $M(\rho, f(z))=\max_\theta|f(\rho e^{i\theta})|$, by obtaining bounds for the maximum modulus of $f_*(z)$ defined by

$$f_*(\xi)=f(\xi)2^{n-m}\prod_1^n g(\xi, b_\nu)/\prod_1^m g(\xi, a_\mu),\quad g(\xi, z)=\frac{z-\xi|z|}{1-\bar{\xi}z},$$

where a_μ, b_ν are the zeros and poles respectively of $f(z)$ in $|(z-\xi)/(1-\bar{\xi}z)|\leqq\frac{1}{2}$. The bounds for $M(\rho, f_*(z))$ are obtained in terms of ρ, $|f_*(0)|$, E and $p(\rho)$ only. When $f(z)$ is regular it is to be noted that $|f(z)|\leqq|f_*(z)|$.

In chapter I, E consists of 0, 1, ∞. Let

$$N_0=\sum_1^\infty(1-|d_\nu|)<\infty$$

where d_ν takes on all values for which $f(d_\nu)=0$, 1 or ∞ in $|z|<1$. Then the author obtains

$$\log M(\rho, f_*(z))<\frac{1}{1-\rho}[(1+\rho)\log^+|f_*(0)|$$
$$+A\rho(\log^+\log^+|f_*(0)|+N_0+1)],$$

which together with four other theorems are derived after obtaining an analogous bound for $\mathrm{Re}\,[f_*'(0)/f_*(0)]$. Extensive calculations are necessary. One of these theorems gives a generalization to functions $f(z)$, such that the equations $f(z)=0$, $\phi(z)$, ∞ have at most $p(\rho)$ roots in $|z|<\rho$ when $\int p(\rho)d\rho<\infty$ and where $\phi(z)$ is meromorphic in $|z|<1$ with at most $p(\rho)$ poles and zeros.

In chapter II $p(\rho)$ is at first taken to be a constant positive integer for $0<\rho<1$. Let r_n be a sequence of real numbers such that $r_0=0$, $r_n<r_{n+1}\to\infty$, $S=\sum_1^\infty(\log r_{n+1}/r_n)^2<\infty$. For each r_n, let $f(z)$, meromorphic in $|z|<1$, either take on some value on $|w|=r_n$ at most $p-1$ times, or take on each value on $|w|=r_n$ at most p times. Then

$$M(\rho, f_*(z))<A(p)\{|f_*(0)|+r_1\}e^{8/p+1}(1-\rho)^{-2p}.$$

This is an extension of a theorem of M. L. Cartwright [Math. Ann. 111, 98–118 (1935)] and an earlier theorem of the author [Proc. London Math. Soc. (2) 51, 450–473 (1949); these Rev. 11, 22].

The second part of chapter II deals with the more general problem, when $p(\rho)$ is unbounded. If E contains only 0, 1, ∞, or is bounded then

$$(1)\qquad \log M(\rho, f_*(z))=\frac{O(1)}{1-\rho}\int_0^{\rho_*}[1+p(r)]dr$$

where $\rho_*=(1+2\rho)/(2+\rho)$. On the other hand, if E contains the whole plane,

$$(2)\qquad \log M(\rho, f_*(z))=O\left\{\int_0^{\rho_*}\frac{1+p(r)}{1-r}dr\right\}.$$

Examples are given to show that when $p(\rho)=(1-\rho)^{-\alpha}$ the order of magnitude in (1) is best possible for $0\leqq\alpha<\infty$ while in (2) it is best possible for $0\leqq\alpha\leqq 1$. It is also shown that if E consists of a sequence w_n for which $1<|w_{n+1}/w_n|<c$, $w_n\to\infty$ then (2) again holds. If $\int_0^1 p(\rho)d\rho<\infty$, it is shown that always

$$\limsup_{\rho\to 1}(1-\rho)\log M(\rho, f_*(z))=0,$$

with further results of this nature.

Chapter III is devoted to converse examples to the results of chapter II, when $p(\rho)=(1-\rho)^{-\alpha}$, $0\leqq\alpha<\infty$.

M. S. Robertson (New Brunswick, N. J.).

Urazbaev, B. M. On the argument of the derivative of a univalent star-shaped function. Izvestiya Akad. Nauk Kazah. SSR. 56, Ser. Mat. Meh. 2, 102–121 (1948). (Russian. Kazak summary)

Let $f(z)=z+\sum_{n=1}^\infty a_nz^{kn+1}$ be univalent and starlike in $|z|=1$. The author proves that the extreme values for $\arg f'(z)$ are attained when $f(z)=z/(1-z^ke^{it})^{2/k}$. In the special case $k=1$, this theorem was proved earlier by Stroganov [Trudy Mat. Inst. Steklov. 5, 247–258 (1934)].

A. W. Goodman (Lexington, Ky.).

Goluzin, G. M. On the problem of coefficients of univalent functions. Doklady Akad. Nauk SSSR (N.S.) 81, 721–723 (1951). (Russian)

Let $f(z)=z+\sum_{n=2}^\infty c_nz^n$, and let $c_{k,l}$ be defined by the expansion

$$\frac{z_1-z_2}{f(z_1)-f(z_2)}=\sum_{k,l=1}^\infty c_{k,l}z_1^{k-1}z_2^{l-1}.$$

The author obtains two conditions, each necessary and sufficient for $f(z)$ to be regular and univalent in $|z|<1$, namely (I) $|c_n|<en$, $n=2, 3, \cdots$, and $|c_{k,l}|<4e(kl)^{1/2}$, $k, l, =1, 2, \cdots$; and (II) $\sum_{k,l=1}^\infty(|c_k|+|c_{k,l}|)/k^3l^3<14e$.

A. W. Goodman (Lexington, Ky.).

✓ ★**Schaeffer, A. C., and Spencer, D. C. Coefficient regions for schlicht functions.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 224–232. Amer. Math. Soc., Providence, R. I., 1952.

This article is a survey of the various methods of attack on the coefficient problem for functions

$$f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots,$$

regular and schlicht in $|z|<1$. To each function of this class there exists a point $(a_2, a_3, \cdots, a_n)$ of a $(2n-2)$-real-dimensional region V_n. The problem consists in finding the precise region V_n.

The authors briefly sketch the earlier classical approaches to the problem which were developed by Gronwall, Faber, Bieberbach, Pick and others in the period 1914–1916. This

led to the more penetrating attack of Löwner [see Math. Ann. **89**, 103–121 (1923)] who expressed each point of V_n in terms of a curve in P_n, the region of points $(c_1, c_2, \cdots, c_{n-1})$ belonging to functions $p(z)=1+2\sum_1^\infty c_n z^n$ of positive real part in $|z|<1$. Here, however, the representation is not one involving only a finite number of parameters, a more desirable goal. The work of Rogosinski [see Math. Z. **35**, 93–121 (1932)] on typically-real functions determined the region T_n which is the convex closure of the region R_n belonging to schlicht functions with real coefficients. The problem received another significant spurt with the characterization of V_n by Peschl [see J. Reine Angew. Math. **176**, 61–94 (1936)] by means of certain tangent cones applied to a region U_n related to V_n in a one-to-one analytic fashion. A short time later Grunsky [see Math. Z. **45**, 29–61 (1939)] was able to characterize V_n by means of an infinite set of inequalities.

The greater part of the survey is devoted to the long strides which have been made in recent years by the use of the variational methods. The first application in this direction was made by Schiffer [see Proc. London Math. Soc. (2) **44**, 432–449 (1938)] who with the present authors and their students have made many significant advances. In particular, it has been possible to express V_n in terms of a finite number of parameters. Most of these results have already been reviewed and are included in detail elsewhere [see A. C. Schaeffer and D. C. Spencer, Coefficient regions for schlicht functions, Amer. Math. Soc. Colloq. Publ., vol. 35, New York, 1950; these Rev. **12**, 326]. For more recent publications see also J. A. Jenkins and D. C. Spencer [Ann. of Math. (2) **53**, 4–35 (1951); these Rev. **12**, 400], M. Schiffer and D. C. Spencer [ibid. **52**, 362–402 (1950); these Rev. **12**, 171], and S. Bergman and M. Schiffer [Compositio Math. **8**, 205–249 (1951); these Rev. **12**, 602].

M. S. Robertson (New Brunswick, N. J.).

★Schiffer, Menahem. Variational methods in the theory of conformal mapping. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 233–240. Amer. Math. Soc., Providence, R. I., 1952.

L'auteur part de la formule d'Hadamard

$$\delta g(z, \zeta)=-\frac{1}{2\pi}\int_C \frac{\partial g(z, t)}{\partial \eta_t}\frac{\partial g(t, \zeta)}{\partial \eta_t}\delta n_t ds_t$$

qui donne la variation de la fonction de Green d'un domaine D de frontière C variable. Formules analogues pour les fonctions $K(z, \bar{\zeta})=-2\pi^{-1}\partial^2 g(z, \zeta)/\partial z\partial\bar{\zeta}$ (noyau de Bergman) et $L(z, \zeta)=-2\pi^{-1}\partial^2 g(z, \zeta)/\partial z\partial\zeta$. Formule plus générale exprimant la différence $K(z, \bar{\zeta})-K_0(z, \bar{\zeta})$ des fonctions K et K_0 relatives à deux domaines non infiniment voisins D, $D_0\supset D$. Pour éviter les difficultés dues aux irrégularités possibles de C, l'auteur étudie d'autre part la variation de $g(z, \zeta)$ dans une déformation infinitésimale $z^*=z+e^{i\varphi}\rho^2/(z-z_0)$ $(z_0 \varepsilon D)$ définie dans tout le plan. Enfin il généralise la formule de Julia [variation de la fonction $f(z)$ qui représente conformément le cercle $D(|z|<1)$ sur un domaine variable Δ avec $f(0)=0$] au cas où D est un domaine quelconque, sous la forme:

$$\delta f(z)=f'(z)\frac{1}{2\pi i}\int_\Gamma n(z, \zeta)\frac{\overline{\delta\omega d\omega}}{[f'(\zeta)]^2} \qquad (\omega=f(\zeta)).$$

Étude de la fonction $n(z, \zeta)$. Applications possibles.

J. Lelong (Lille).

Ozawa, Mitsuru. On an application of Hadamard's variational method to conformal mapping. Kōdai Math. Sem. Rep. **1951**, 41–42 (1951).

Let D_3 be a domain bounded by three curves, Γ_1, Γ_2, and Γ_3, let z_1 and z_2 be points of Γ_1, and let Γ' and Γ'' be the arcs of Γ between z_1 and z_2. Denote by $\Phi(z)$ a conformal mapping of D_3 onto the strip $0<\operatorname{Re}\Phi<1$, with two parallel slits, such that $\operatorname{Im}\{\Phi(z_1)\}=-\infty$, $\operatorname{Im}\{\Phi(z_2)\}=+\infty$, and such that Γ' corresponds to $\operatorname{Re}\Phi=1$. It is shown that if the boundary arc Γ'' is shifted so that it includes a larger domain D_3^* with map function $\Phi^*(z)$, then on Γ_2 and Γ_3 $\operatorname{Re}\{\Phi^*(z)\}\geqq\operatorname{Re}\{\Phi(z)\}$.

P. R. Garabedian.

Ozawa, Mitsuru. On classification of the function-theoretic null-sets on Riemann surfaces of infinite genus. Kōdai Math. Sem. Rep. **1951**, 43–44 (1951).

Consider an m-sheeted Riemann surface R with branch points lying above the x-axis. On R, let E be a closed point set. The problem is to decide when every single-valued analytic bounded function on $R-E$ can be continued analytically on E. The author takes m replicas of E (selected in a suitable manner) and considers the projection P on the complex plane F of the intersection of these copies. Then E is removable for bounded functions on $R-E$ if and only if P is removable for bounded functions on $F-P$. The same is true for analytic functions with a finite Dirichlet integral. In the proofs use is made of Myrberg's reasoning [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **58** (1949); these Rev. **10**, 441].

L. Sario.

Nagura, Shohei. Kernel functions on Riemann surfaces. Kōdai Math. Sem. Rep. **1951**, 73–76 (1951).

The author sets up various kernel functions on an abstract Riemann surface F and formulates corresponding conditions in terms of these kernels for the ideal boundary of F to be a null-boundary.

P. R. Garabedian.

★Radojčić, M. Sur le discernement des types des surfaces de Riemann. Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 163–167. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

Exposition élémentaire de deux critères, obtenus par l'auteur, concernant le type d'une surface de Riemann ouverte et simplement connexe [voir Radojčić, Glas Srpske Akad. Nauka **175**, 237–248 (1937); Acad. Serbe Sci. Publ. Inst. Math. **3**, 25–52, 305–306 (1950); ces Rev. **12**, 602].

From the author's summary.

★Nevanlinna, Rolf. Surfaces de Riemann ouvertes. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 247–252. Amer. Math. Soc., Providence, R. I., 1952.

Expository paper giving the present stage of the classification theory of open Riemann surfaces.

L. V. Ahlfors (Cambridge, Mass.).

Heins, Maurice. Interior mapping of an orientable surface into S^2. Proc. Amer. Math. Soc. **2**, 951–952 (1951).

Démonstration simple et élégante de cette proposition que le rapporteur avait établie, dans ses Leçons sur les principes topologiques de la théorie des fonctions analytiques [Gauthier-Villars, Paris, 1938], par un raisonnement non suffisamment explicite: toute surface orientable peut être prise pour type topologique d'une surface de Riemann. La

démonstration de l'auteur, montre, de plus, qu'il existe un recouvrement de la sphère par la surface donnée, tel que les points de ramification se projettent en trois points seulement. *S. Stoïlow* (Bucarest).

√ ★**Cartan, Henri. Problèmes globaux dans la théorie des fonctions analytiques de plusieurs variables complexes.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 152–164. Amer. Math. Soc., Providence, R. I., 1952.

The author surveys, and adds to, very important results, mostly due to himself and also K. Oka, on the impact of the concept of an ideal of analytic functions on the question of maximal continuation of analytic varieties, or rather subvarieties, with algebroid singularities, on the so-called problems of Cousin, and also on such questions as the construction of differential forms with prescribed periods if the space is not compact. We will only single out the theorem that if a variety is a so-called domain of holomorphy, then any subvariety which locally is the zero-set of an ideal, and, which is decisive, if these ideals are locally coherent in the definition of the author, is such a zero-set of a global ideal in its entirety.

The author makes it clear, and it ought to be emphasized, that for most results stated the underlying general space is general only to the extent of not being an ordinary domain in the z-variables, but that it is usually presupposed as being spread out over the ordinary space multi-sheetedly. For more than one variable, spaces of this kind are a much more particular subclass of all complex spaces than the very contrary phenomenon of one complex variable would at first lead one to expect. *S. Bochner* (Princeton, N. J.).

Wray, Joe W. Non-analytic functions of a complex variable representable by Lebesgue-Stieltjes integrals with a Cauchy kernel. Abstract of a Thesis, University of Illinois, Urbana, Ill., 1952. 2+i pp.

Theory of Series

Vanderburg, B. Certain linear combinations of Hausdorff summability methods. Trans. Amer. Math. Soc. **71**, 466–477 (1951).

Let C^α and H^α denote the Cesàro and Hölder methods of summability (order α) which are known to be equivalent when Re $\alpha > -1$. The method

$$\frac{C^\alpha - \Gamma(\alpha+1)H^\alpha}{1-\Gamma(\alpha+1)}$$

is shown to include $H^{\alpha+1}$, and both are equivalent for real $\alpha > -1$, $\alpha \neq 0, 1$. Again

$$\frac{\Gamma(\alpha+\beta+1)C^\alpha C^\beta - \Gamma(\alpha+1)\Gamma(\beta+1)C^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)-\Gamma(\alpha+1)\Gamma(\beta+1)}$$

includes $C^{\alpha+\beta+1}$ and the two methods are equivalent when α, β and $\alpha+\beta$ are >-1 and $\alpha, \beta \neq 0$. A similar result is proved for a certain combination of Riesz-methods $C_\alpha{}^\lambda$, of order α and type λ. The proofs are based on the investigation of the respective Mellin transforms of these methods. The inclusion proofs are simple, but the equivalence requires an elaborate discussion and, in part, numerical computation. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Bosanquet, L. S. Note on a theorem of M. Riesz. Proc. London Math. Soc. (3) **1**, 453–461 (1951).

Let $A_n{}^\kappa = (\kappa+1)\cdots(\kappa+n)/n! \sim n^\kappa/\Gamma(\kappa+1)$ be the Cesàro numbers, and let $S^\kappa(\sum_0^n a_\nu)$ denote the Cesàro sums of the series $\sum a_\nu$. The author begins by giving a simplified version of the following result, which is essentially known [see, e.g., Hardy and Riesz, The general theory of Dirichlet's series, Cambridge Univ. Press, 1915; esp. Theorem 48]: If $\kappa > 1$, $0 < \sigma < \kappa+1$, τ real, and if $\sum a_n$ is summable (C, κ) then $\sum (n+1)^{-\sigma-i\tau} a_n$ is summable $(C, \kappa-\sigma)$. The method used is subsequently applied to a refinement of this theorem, namely: If $\kappa > -1$, σ is real, and $\sum a_n$ is summable (C, κ), then there are constants $c_0, c_1, \cdots$ such that

$$S^\kappa(\textstyle\sum_0^n (\nu+1)^{-\sigma} a_\nu) = \sum_{j=0}^{[\sigma]} c_j A_n{}^{\kappa-j} + o(n^{\kappa-\sigma}).$$

The result holds if, on the left, σ is replaced by $\sigma+i\tau$, and $\sigma \neq 0, 1, \cdots$. If $\sigma = 0, 1, \cdots$ the result is false, except when either $\tau = 0$ or κ is an integer and $\sigma = \kappa+1, \kappa+2, \cdots$. *A. Zygmund* (Chicago, Ill.).

Korenblyum, B. I. Theorems of Tauberian type for a class of Dirichlet series. Doklady Akad. Nauk SSSR (N.S.) **81**, 725–727 (1951). (Russian)

A generalization of the Hardy-Littlewood "high indices theorem" with an unusual Tauberian condition is announced: If the positive numbers λ_n for some non-negative integer r satisfy the condition $\lambda_{n+r+1}/\lambda_n \geqq k > 1$, then Abel's $A(\lambda_n)$-summability of a series $\sum u_n$ implies its Riesz (R, λ_n, r)-summability (Hardy-Littlewood's theorem is obtained by taking $r=0$). Moreover, with a constant $M = M(r, k)$,

$$\sup_{\tau>0}\left|\sum_{\lambda_n \leqq \tau}\left(1-\frac{\lambda_n}{\tau}\right)^r u_n\right| \leqq M \sup_{x>0}\left|\sum_{n=1}^\infty u_n e^{-\lambda_n x}\right|.$$

If $\liminf_{n\to\infty} (\lambda_{n+r+1}/\lambda_n) = 1$, there is an $(R, \lambda_n, r+1)$-summable series whose (R, λ_n, r)-means are not bounded. *G. G. Lorentz* (Toronto, Ont.).

Garten, V. Über Taubersche Konstanten bei Cesàroschen Mittelbildungen. Comment. Math. Helv. **25**, 311–335 (1951).

Let k be a positive integer and let α be a positive parameter. A formula, involving $\int_0^\alpha \{1-(1-t)^k\} t^{-1} dt$ and similar integrals, is given for the least function $C_k{}^*(\alpha)$ having the following property. Let $\sum a_n$ be a series of complex terms satisfying the Tauberian condition $\limsup |na_n| < \infty$, and let $s_0, s_1, \cdots$ and $c_0{}^{(k)}, c_1{}^{(k)}, \cdots$ denote its sequence of partial sums and its Cesàro transform of order k. Then

$$(*) \qquad \limsup |c_p{}^{(k)} - s_n| \leqq C_k{}^*(\alpha) \limsup_{m\to\infty} |ma_m|$$

when p and n become infinite in such a way that $p = [n/\alpha]$ or $n = [\alpha p]$. For each k, the function $C_k{}^*(\alpha)$ is a minimum and the estimate in (*) is best when $\alpha = \alpha_k{}^* = 1-2^{-1/k}$. The sequence $t_k{}^* = C_k{}^*(\alpha_k{}^*)$ has elements $t_1{}^* = \log 2 = .693147$, $t_2{}^* = .813732$, $t_3{}^* = .86106$ and is increasing. Moreover, as $k\to\infty$, $t_k{}^* \to t^* = .9680448$ where t^* is a fundamental Tauberian constant governing partial sums and Abel power series transforms of Tauberian series. Similar results are obtained for series satisfying the weaker Tauberian condition $\limsup n^{-1}|\sum_{k=1}^n ka_k| < \infty$. *R. P. Agnew.*

Braunbek, Werner. Ein der WKB-Methode verwandtes Verfahren zur Entwicklung von Wellenfeldern. Z. Naturforschung **6a**, 672–676 (1951).

It is shown that an asymptotic expansion of the complex amplitude $h \exp(iks)$ of a wave field is possible in powers

of $(ik)^{-1}$, if complex values of h are admitted. This method is compared to the usual WKB-method, in which complex values of s are admitted; the homogeneous cylinder-wave is treated as an example, and it is shown that both methods lead to the same asymptotic expansion. *T. E. Hull.*

van Veen, S. C. Asymptotic expansion of the generalized Bernoulli numbers $B_n^{(n-1)}$ for large values of n (n integer). Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 335–341 (1951).

The generalized Bernoulli numbers $B_n^{(z)}$ are defined by

$$\left(\frac{t}{e^t-1}\right)^z=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(z)},\qquad |t|<2\pi.$$

The author shows that

$$\left\{\frac{\log(1+t)}{t}\right\}^z=z\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{B_n^{(z+n)}}{z+n},\qquad |t|<1,$$

and then considers the case $z=-1$. By contour integration and an appropriate change of variable, he obtains the asymptotic expansion

$$\frac{(-1)^nB_n^{(n-1)}}{n!(n-1)}=\sum_{l=1}^{m}\frac{\sin l\alpha}{\sin\alpha}\frac{E_l(n)}{\rho^{l+1}}+\frac{\theta(1+|\cos m\alpha|)}{2\sin^2\alpha}\frac{E_{m+1}(n)}{\rho^{m+2}}.$$

Here $|\theta|\leqq 1$, $\tan\alpha=\pi/S_1$, $S_1=\sum_{r=1}^{n-2}1/r$, $\rho=(\pi^2+S_1^2)^{1/2}$, $E_1(n)=(n-1)^{-1}$, $E_2(n)=0$, and for $l>2$, $E_l(n)$ can be determined recursively. The first few are

$$E_3(n)=\frac{1}{n-1}\left(\frac{\pi^2}{3}-S_2\right),$$

$$E_4(n)=\frac{1}{n-1}(-2S_3),$$

$$E_5(n)=\frac{1}{n-1}\left(\frac{7\pi^4}{15}-2\pi^2S_2+3S_2^2-6S_4\right),$$

$$E_6(n)=\frac{1}{n-1}\left(-20\frac{\pi^2S_3}{3}+20S_2S_3-24S_5\right),$$

where $S_k=\sum_{r=1}^{n-2}r^{-k}$. *N. J. Fine* (Philadelphia, Pa.).

Fourier Series and Generalizations, Integral Transforms

Boas, R. P., Jr. Integrability of trigonometric series. I. Duke Math. J. **18**, 787–793 (1951).

Let $\sum a_n$ be an absolutely convergent series of real numbers and

$$(\gamma)\quad g(x)=\sum_1^{\infty}a_n\sin nx,\qquad (\varphi)\quad f(x)=\tfrac{1}{2}a_0+\sum_1^{\infty}a_n\cos nx.$$

It is shown that the Cauchy limit $\int_{\to 0}x^{-1}g(x)dx$ always exists, while necessary and sufficient conditions for the existence of (i) $\int_{\to 0}x^{-1}f(x)dx$ are that $\frac{1}{2}a_0+\sum_1^{\infty}a_n=0$ and that $\sum_1^{\infty}n^{-1}\sum_{k=n+1}^{\infty}a_k$ converges. If the a_k are ultimately of one sign, the latter condition is equivalent to the convergence of $\sum a_k\log k$; (i) is then a Lebesgue integral. If one supposes that $\sum|\Delta a_k|<\infty$, $a_k\to 0$, instead of supposing that $\sum|a_k|<\infty$, then it is known that (γ) and (φ) converge uniformly in the interior of $(0,2\pi)$ and that $\int_{\to 0}^{\pi}f(x)dx$ always exists [see Szidon, Math. Z. **10**, 121–127 (1921)]; it is now shown that $\int_{\to 0}^{\pi}g(x)dx$ exists if and only if $\sum n^{-1}a_n$ converges. In the course of the proof the following proposition is established, which is interesting in itself: If $\sum|\Delta(nb_n)|<\infty$ (in particular if $nb_n\downarrow 0$), then

$$\varphi(x)=\sum_1^{\infty}b_n\cos nx$$

converges for $0<x<2\pi$, and $\lim_{x\to+0}\varphi(x)$ exists if and only if $\sum_1^{\infty}b_n$ converges; when this is the case, $\varphi(x)$ is continuous at 0. *B. Sz.-Nagy* (Szeged).

Gál, István Sándor. Sur la convergence d'interpolations linéaires. III. Fonctions continues. C. R. Acad. Sci. Paris **233**, 1001–1003 (1951).

The results of part II [same C. R. **233**, 347–350 (1951); these Rev. **13**, 229] are extended to the case when the interpolated function is continuous not only at individual points but over the whole domain of definition. *A. Zygmund* (Chicago, Ill.).

Sz.-Nagy, Béla. Sur l'ordre de l'approximation d'une fonction par son intégrale de Poisson. Acta Math. Acad. Sci. Hungar. **1**, 183–188 (1950). (French. Russian summary)

Let $\mathrm{Lip}_1\,\alpha$ denote the class of functions $f(x)$ of period 2π satisfying the inequality $|f(x_2)-f(x_1)|\leqq|x_2-x_1|^{\alpha}$ $(0<\alpha\leqq 1)$. Let $\bar f(x)$ be the function conjugate to $f(x)$, and $\bar f(r,x)$ the Poisson integral of $\bar f(x)$. Finally, let

$$v(r,\alpha)=\sup_f\max_x|\bar f(r,x)-\bar f(x)|$$

for all $f\epsilon\mathrm{Lip}_1\,\alpha$. The author shows that

$$v(r,\alpha)=2^{\alpha-1}\operatorname{cosec}\tfrac{1}{2}\pi\alpha(1-r)^{\alpha}+O(1-r)^{\alpha+1}\quad(0<\alpha\leqq 1).$$

[Compare also, I. P. Natanson, Doklady Akad. Nauk SSSR (N.S.) **72**, 11–14 (1950); these Rev. **11**, 655.] *A. Zygmund* (Chicago, Ill.).

Natanson, I. P. On a class of singular double integrals. Doklady Akad. Nauk SSSR (N.S.) **81**, 737–739 (1951). (Russian)

Let $\Phi(u)$ and $\Psi(v)$ be two continuous and even functions of period 2π, such that $\int_{-\pi}^{\pi}\Phi_n du=\int_{-\pi}^{\pi}\Psi_m dv=1$. We suppose that the numbers $\lambda_n=\int_0^{\pi}u\Phi_n du$ and $\mu_m=\int_0^{\pi}v\Psi_m dv$ tend to 0 as $m,n\to\infty$. Let $f(x,y)$ be any continuous function of period 2π in x and y, and let $\omega(\lambda,\mu)$ be the modulus of continuity of f, that is $\omega(\lambda,\mu)=\sup|f(x+h,y+k)-f(x,y)|$ for all x, y, $|h|\leqq\lambda$, $|k|\leqq\mu$. Let

$$f_{nm}(x,y)=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(u,v)\Phi_n(u-x)\Psi_m(v-y)dudv.$$

The author shows that $|f_{nm}(x,y)-f(x,y)|\leqq 5\omega(\lambda_n,\mu_m)$. *A. Zygmund* (Chicago, Ill.).

Chandrasekharan, K. Fourier series, lattice points and Watson transforms. Math. Student **19**, 1–11 (1951).

A review, without proofs, of problems and results obtained recently in the theory of multiple Fourier series. The discussion is mainly confined to problems connected with the notion of spherical convergence and summability of such series. *A. Zygmund* (Chicago, Ill.).

Yano, Shigeki. On approximation by Walsh functions. Proc. Amer. Math. Soc. **2**, 962–967 (1951).

Let $f(x)\epsilon\mathrm{Lip}\,\alpha$; $0<\alpha<1$, and let $\sigma_n(x;f)$ be the $(C,1)$-mean of the Walsh-Fourier series of f. The reviewer [Trans. Amer. Math. Soc. **65**, 372–414 (1949); these Rev.

11, 352] asked whether, as for trigonometric Fourier series, $\sigma_n(x; f) - f(x) = O(n^{-\alpha})$. By explicitly evaluating the kernel, the author answers this question in the affirmative. More generally, he shows that $\sigma_n^{(\beta)}(x; f) - f(x) = O(n^{-\alpha})$, where $\sigma_n^{(\beta)}$ is the (C, β)-mean, for any $\beta > \alpha$. Finally, he states the corresponding result for the integrated Lipschitz condition: If $f \epsilon \mathrm{Lip}\,(\alpha, p)$, $0 < \alpha < 1$, $p \geqq 1$, then for any $\beta > \alpha$,

$$\left(\int_0^1 |\sigma_n^{(\beta)}(x; f) - f(x)|^p dx \right)^{1/p} = O(n^{-\alpha}).$$

N. J. Fine (Philadelphia, Pa.).

Yano, Shigeki. On Walsh-Fourier series. Tôhoku Math. J. (2) 3, 223–242 (1951).

Let $\{\psi_n(x)\}$ denote the Walsh functions. The author proves a number of theorems concerning Cesàro summability, convergence, special series, and convergence factors for Walsh series $\sum c_n \psi_n(x)$, most of them for Walsh-Fourier series (WFS). The results and methods are almost completely analogous to those in trigonometric series. Th. 1. If $f(x) \sim \sum c_n \psi_n(x)$, then the series $\sum c_n \psi_n(x)/(n+1)^\alpha$, $0 < \alpha < 1$, is summable $(C, -\alpha)$ almost everywhere. Th. 2 is the direct generalization to multiple series. Th. 3. If $\lim_{x \to x_0} f(x) = s$ exists, then the $(C, 1)$-sums $\sigma_n(x; f) \to s$. [The author attributes Th. 3 to Walsh [Amer. J. Math. 45, 5–24 (1923)]. If the reviewer remembers correctly, Walsh assumed continuity in an interval, as did the reviewer [Trans. Amer. Math. Soc. 65, 372–414 (1949); these Rev. 11, 352]. The author's proof follows closely that of the reviewer, except that he uses his explicit evaluation of the Fejér kernel. See the preceding review.] Th. 4. Under the assumption of Th. 3, the (C, α)-sums $\sigma_n^{(\alpha)}(x; f) \to s$, for $\alpha > 0$. Th. 5. If $f(x)$ satisfies

$$\frac{1}{t} \int_0^t |f(x+u) - f(x)| du = O\left(\frac{1}{\log |t|^{-1}} \right)$$

for every point x in a set E of positive measure, then the WFS of f converges for almost all x in E. Th. 6. For $1 \leqq p \leqq 2$, if $f(x)$ satisfies

$$\int_0^1 \int_0^1 \frac{|f(x+t) - f(x-t)|^p}{t} dx dt < \infty,$$

then the WFS of f converges almost everywhere. Th. 7. If $c_n \to 0$ and $\{c_n\}$ is quasi-convex, the series $\sum c_n \psi_n(x)$ converges, save for $x = 0$, to an integrable function $f(x)$, and is the WFS of f. Th. 8. If $c(x)$, $x \geqq 0$, is a positive, convex function tending to 0 as $x \to \infty$, and if for $c_k = c(k)$, $k(c_k - c_{k+1})$ is non-increasing, and if $\sum c_n = \infty$, then $f(x) \sim \int_0^{1/x} t |c'(t)| dt$ as $x \to +0$, where $f(x) = \sum c_n \psi_n(x)$. Th. 9. Let $f(x) \sim \sum c_n \psi_n(x)$, and define

$$s_n^*(x) = \sum_{k=0}^{n-1} \frac{c_k}{\log (k+2)} \psi_k(x),$$

$$s_n^{**}(x) = \sum_{k=0}^{n-1} \frac{c_k}{\{\log (k+2)\}^{\frac{1}{2}}} \psi_k(x).$$

1) If $f(x) \epsilon L^2$, then

$$\int_0^1 \sup_n |s_n^{**}(x)|^2 dx \leqq A \int_0^1 |f(x)|^2 dx,$$

where A is an absolute constant.

2) If $f(x) \log (1 + f^2(x)) \epsilon L$, then

$$\int_0^1 \sup_n |s_n^*(x)| dx \leqq A \int_0^1 |f(x)| \log (1 + f^2(x)) dx + B,$$

where A and B are absolute constants.

3) If $f(x) \epsilon L$, then for $0 < r < 1$,

$$\left\{ \int_0^1 \sup_n |s_n^*(x)|^r dx \right\}^{1/r} \leqq A_r \int_0^1 |f(x)| dx,$$

where A_r depends only on r. Thus, for $p = 1$ and $p = 2$, $\{1/\log (n+2)\}^{1/p}$ is a convergence factor for the WFS of functions in L^p. This was proved by Paley for $1 \leqq p \leqq 2$ [Proc. London Math. Soc. (2) 34, 241–264, 265–279 (1932)].

N. J. Fine (Philadelphia, Pa.).

Hirschman, I. I., Jr. On approximation by non-dense sets of translates. Amer. J. Math. 73, 773–778 (1951).

L. Schwartz hat gezeigt, dass in $L_2(-\infty, \infty)$ und für alle reellen x definierten Funktionen $f(x)$,

$$\|f(x)\| = \left[\int_{-\infty}^{+\infty} |f(x)|^2 dx \right]^{\frac{1}{2}} < \infty,$$

sich für fast alle x durch eine Reihe

$$f(x) = \sum_{n=1}^{\infty} b_n e^{\frac{1}{2}(t_n - x)} \exp\left[-e^{(t_n - x)} \right]$$

darstellen lassen, wenn

$$\|f(x) - \sum_{n=1}^{N} a_n e^{\frac{1}{2}(t_n - x)} \exp\left[-e^{(t_n - x)} \right]\| \leqq \epsilon$$

für ein geeignetes N und die t_i gewisse Bedingungen erfüllen. Der Verf. verallgemeinert diesen Satz für den Fall, dass

$$\phi(x) \epsilon L_2(-\infty, \infty), \quad \psi(t) \epsilon L_2(-\infty, \infty)$$

und

$$\phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{itx} \psi(t) dt, \quad \psi(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-itx} \phi(x) dx.$$

Er zeigt, dass unter der Voraussetzung, dass

$$\|f(x) - \sum_{n=1}^{N} a_n \phi_n(t_n - x)\| < \epsilon,$$

$f(x)$ sich für fast alle Werte von x bei geeigneter Wahl der t_n in der Form darstellen lässt

$$f(x) = \sum_{n=1}^{\infty} b_n \phi_n(t_n - x).$$

W. Saxer (Zürich).

*✶Bohr, Harald. A survey of the different proofs of the main theorems in the theory of almost periodic functions.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 339–348. Amer. Math. Soc., Providence, R. I., 1952.

The main definitions and results in the theory of almost periodic functions are summarized and the different proofs of the Parseval equation, the unicity theorem, and the approximation theorem, due to the author, de la Vallée Poussin, Weyl, Wiener, Bochner, Bogolioùboff, and others, are described.

B. Jessen (Copenhagen).

Jenkins, James A. Generalization of a theorem of Mandelbrojt. Amer. J. Math. 73, 807–812 (1951).

L'auteur démontre le théorème suivant: soient f et φ deux fonctions appartenant à $L^2(-\infty, \infty)$, transformées de Fourier l'une de l'autre. Posons

$$U(r) = \pi^{-1} \int_{-r}^{r} (1 + x^2) \log |\varphi(x)| dx,$$

et supposons qu'il existe une fonction non décroisante $V(y)$ $(y>0)$ telle que

$$(1) \qquad \int_{V(y)}^{\infty} e^{ty}|f(t)|\,dt = O(e^{yV(y)}) \qquad (y\to\infty).$$

Si $\lim_{r\to\infty} U(r) = -\infty$, et si $\liminf_{r\to\infty} [V(r)+U(r)] = -\infty$, les fonctions f et φ sont nulles presque partout. Soit, en particulier, f une fonction indéfiniment dérivable pour $-\infty<x<\infty$, et supposons que

$$|f^{(n)}(x)| \leqq Ak^n M_n \quad (-\infty<x<\infty,\ n\geqq 0).$$

Posons

$$W(r, h) = 2\pi^{-1}\int^{r}(1+x^2)^{-1}\log T(x/h),$$

où $T(r) = $ borne $\sup_{n\geq 1} r^n M_n^{-1}$, et où h est une constante, $h>k$. Si (2) $\lim M_n^{1/n} = \infty$, si

$$\liminf_{r\to\infty}[V(r)-W(r, h)] = -\infty,$$

et si (1) est satisfait, la fonction f est identiquement nulle. [L'auteur introduit la suite $\{M_n^c\}$, la régularisée convexe de $\{M_n\}$, et suppose que (3) $\lim (M_n^c)^{1/n} = \infty$. Ceci parait inutile, car $\{M_n^c\}$ n'existe que si (1) a lieu, et alors les conditions (2) et (3) sont équivalentes.] *S. Mandelbrojt.*

Barrucand, Pierre. Sur la transformation de Stieltjes d'une série de Taylor et son itération. C. R. Acad. Sci. Paris **233**, 1562–1563 (1951).

Let $\phi(s)$, $\xi(s)$ be analytic functions for $\operatorname{Re} s \geqq -\epsilon$, and let $f(x) = \sum_{n=0}^{\infty}\phi(n)(-x)^n$, $g(x) = \sum_{n=0}^{\infty}\xi(n)(-x)^n$. The author establishes various formal identities of which the following are typical:

$$\int_0^{\infty}\frac{f(y)}{x+y}dy = -\log x f(-x) - \sum_{n=0}^{\infty}\phi'(n)x^n,$$

$$\int_0^{\infty} f(xy)g\left(\frac{1}{y}\right)dy = -\log x\sum_{n=0}^{\infty}\phi(n)\xi(n)x^n - \sum_{n=0}^{\infty}[\phi(n)\xi'(n)+\phi'(n)\xi(n)]x^n.$$

I. I. Hirschman, Jr. (St. Louis, Mo.).

San Juan, Ricardo. Caractérisations fonctionnelles des transformations de Laplace. Portugaliae Math. **10**, 115–120 (1951).

Let D be any non-vacuous set in the complex plane. The author shows that if $\mathfrak{L}_s$ is for $s\epsilon D$ a bounded linear functional on $L_1(0, 1)$ and if

$$\mathfrak{L}_s\left[\int_x^1 \psi(u)\,du\right] = \mathfrak{L}_s[1]\mathfrak{L}_s[x\psi(x)],$$

$$\operatorname{Re}(\mathfrak{L}_s[1])>1, \quad s\epsilon D,$$

then $\mathfrak{L}_s$ is of the form

$$\mathfrak{L}_s[\psi(x)] = \int_0^1 \psi(x)x^{g(s)-1}\,dx,$$

where $\operatorname{Re} g(s)>1$ for $s\epsilon D$. These results are closely related to those given by Doetsch [Math. Nachr. **5**, 219–230 (1951); these Rev. **13**, 127]. *I. I. Hirschman, Jr.*

Delange, Hubert. Sur les singularités des intégrales de Laplace. C. R. Acad. Sci. Paris **233**, 1413–1414 (1951).

Let $\alpha(t)$ be a complex function defined for $t\geqq 0$ and of bounded variation on every finite interval. Let the Laplace-Stieltjes transform of $\alpha(t)$, $f(s) = \int_0^{\infty} e^{-st}d\alpha(t)$, have a finite abscissa of convergence σ_c. The author proves that if there exist a real continuous function $\psi(t)$ and a real constant ϕ, $0\leqq\phi<\frac{1}{2}\pi$, such that the argument of $\int_{t'}^{t''} e^{-i\psi(u)}d\alpha(u)$ does not exceed ϕ in absolute value when t' and t'' are large, and if $|\psi(t')-\psi(t'')| \leqq k|t''-t'|$ for t' and t'' large, then $f(s)$ has at least one singularity on the segment

$$[\sigma_c - ikC(\phi), \sigma_c + ikC(\phi)].$$

Here $C(\phi)$ is a non-decreasing function of ϕ such that $C(0) = 1$ and $C(\phi)$ increases to ∞ as ϕ approaches $\frac{1}{2}\pi$. This theorem contains many earlier results on the singularities of Dirichlet series and power series as special cases.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Levitan, B. M., and Meiman, N. N. On a uniqueness theorem. Doklady Akad. Nauk SSSR (N.S.) **81**, 729–731 (1951). (Russian)

Let $\sigma(\lambda)$ be such that for all real x

$$\int_{-\infty}^{\infty}\cos\lambda^{\frac{1}{2}}x\,d\sigma(\lambda) = 0, \quad \int_{-\infty}^{\infty}|\cos\lambda^{\frac{1}{2}}x|\cdot|d\sigma(\lambda)|<\infty,$$

and such that as $x\to+\infty$

$$\int_{-\infty}^{0}\exp(|\lambda|^{\frac{1}{2}}x)|d\sigma(\lambda)| = O(\exp(\alpha x^2)),$$

where α is some positive constant. Then $\sigma(\lambda) = $ const. This sharpens a previous result of B. M. Levitan [Doklady Akad. Nauk SSSR (N.S.) **76**, 485–488 (1951); these Rev. **12**, 605]. Analogous results are indicated for integrals with the sine-function or with Bessel functions. The result can be used to simplify the proof of eigen-function expansions over $(0, \infty)$. *F. V. Atkinson* (Ibadan).

Polynomials, Polynomial Approximations

Angheluţă, Theodor. The number of roots with positive imaginary parts of an algebraic equation. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 129–136 (1950). (Romanian. Russian and French summaries)

Let a_i $(0\leqq i\leqq n)$ be $n+1$ complex numbers, set

$$F(z) = \sum_{i=0}^{n} a_i z^{n-i}, \quad F_0(z) = \sum_{i=0}^{n}\bar{a}_i z^{n-i},$$

let (*) $F(z) = 0$ have the roots α_ν $(1\leqq\nu\leqq n)$, all distinct, and assume that $F(z)$ is not divisible by any polynomial with real coefficients. The author establishes a sequence whose permanences of sign are equal to the number of roots of (*) with positive imaginary parts. Use is made of the following theorem of Hermite [Oeuvres, tome 1, Gauthier-Villars, Paris, 1905, pp. 397–414; J. Reine Angew. Math. **52**, 39–51 (1856)]. Let

$$\varphi = \sum_{h=1}^{n}\frac{i}{F_0(\alpha_h)F'(\alpha_h)}(x+\alpha_h y+\cdots+\alpha_h^{n-1}u)^2.$$

Then φ has real coefficients and if we reduce it by a real, linear transformation, to a sum of squares, the number of positive squares is equal to the number of roots of (*) with positive imaginary parts. Taking the new variables z,

$(0 \leqq \nu \leqq n-1)$, defined by $\frac{1}{2}\partial\varphi/\partial x = z_0, \cdots, \frac{1}{2}\partial\varphi/\partial u = z_{n-1}$, φ becomes

$$\psi = \sum_{h=1}^{n} \frac{-iF_0(\alpha_h)}{F'(\alpha_h)} \vartheta^2(\alpha_h)$$

where $\vartheta(\alpha_h)$ are linear polynomials in the variables z_ν. Using the expansion $-i(\sum_{i=0}^{n} \bar{a}_i u^i)/(\sum_{i=0}^{n} a_i u^i) = \sum_{\nu=0}^{\infty} c_{\nu-1} u^\nu$, the author shows that $\psi = \sum c_{h+k} y_h y_k$, $h, k = 0, 1, \cdots, n-1$, where $y_h = \sum_{\nu=0}^{n-h-1} a_\nu z_{n-\nu-h-1}$. In order to find the number of positive squares of ψ, use is made of an identity involving determinants, formed with the coefficients a_i, $\bar{a}_i$ and c_ν, obtaining finally the following result: Define the real determinants

$$D_m = \begin{vmatrix} a_0 & a_1 & \cdots & a_{2m-1} \\ i\bar{a}_0 & i\bar{a}_i & \cdots & i\bar{a}_{2m-1} \\ 0 & a_0 & \cdots & a_{2m-2} \\ 0 & i\bar{a}_0 & \cdots & i\bar{a}_{2m-2} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ 0 & 0 & \cdots & i\bar{a}_m \end{vmatrix}$$

and consider the sequence $1, D_1, \cdots, D_n$. The number of its permanences equals the number of roots of (*) with positive imaginary parts. Starting from the equation $(-i)^n F(iz) = 0$, we obtain correspondingly a sequence of determinants $1, D_1', \cdots, D_n'$, such that the number of permanences of this sequence equals the number of roots of (*) with negative imaginary parts. A necessary and sufficient condition that all roots of (*) have negative imaginary parts is, therefore, $D_i > 0$ $(1 \leqq i \leqq n)$. This contains as a particular case a classical theorem of Hurwitz [Math. Ann. **46**, 273–284 (1895) = Mathematische Werke, Bd. 2, Birkhäuser, Basel, 1933, pp. 533–545]. *E. Grosswald.*

Davis, Philip, and Pollak, Henry. On the zeros of total sets of polynomials. Trans. Amer. Math. Soc. **72**, 82–103 (1952).

Soit B un domaine borné simplement connexe du plan complexe z; L_B une classe de fonctions régulière dans B, telle que, pour chaque point de la frontière de B, il existe dans L_B des fonctions ayant une singularité aussi voisine qu'on veut de ce point. Une suite de polynomes

$$p_n(z) = a_n^n z^n + \cdots \quad (a_n^n \neq 0, n = 0, 1, \cdots)$$

est dite totale dans L_B si toute fonction f de L_B possède un développement de la forme $f(z) = \sum_0^\infty b_n p_n(z)$ convergent à l'intérieur de B et si, de plus, les zéros des p_n forment un ensemble borné. Pour l'ensemble dérivé de cet ensemble, on a le théorème fondamental suivant: dans chacun des demi-plans fermés déterminés par la médiatrice du segment joignant deux points de la frontière, il y a au moins un point d'accumulation.

Application aux lemniscates, aux proprietés que doit posséder la frontière d'un domaine pour qu'une suite de polynomes de Tchebycheff y soit totale. Étude du cas où l'ensemble dérivé est fini, chacun des points d'accumulation possédant une densité (positive) d'accumulation. Application à la représentation conforme par l'intermédiaire des polynomes de Faber, propriétés geométriques de certaines courbes de niveau dans le cas où B a une frontière analytique. Généralisation aux domaines multiplement convexes, mais en considérant, au lieu de polynomes, des suites de fractions rationelles introduites par Walsh.

J. Favard (Paris).

Breusch, Robert. On the distribution of the roots of a polynomial with integral coefficients. Proc. Amer. Math. Soc. **2**, 939–941 (1951).

D. H. Lehmer [Ann. of Math. (2) **34**, 461–479 (1933), p. 476] proposed the problem of finding polynomials $f(z) = z^r + a_1 z^{r-1} + \cdots + a_r$ with integral coefficients a_k and $a_r = \pm 1$ such that $1 < \Omega(f) < 1 + \epsilon$, where $\Omega(f)$ is the modulus of the product of all the zeros of $f(z)$ which lie outside the unit circle. In the present paper, the author shows that, if $f(z)$ is a nonreciprocal polynomial, then $\Omega(f) \geqq 1.179$. His proof involves the resultant $R(f, g)$ of $f(z)$ with zeros z_n and the polynomial $g(z) = \pm z^r f(1/z)$ with zeros $z_n^* = 1/\bar{z}_n$. Setting

$$P_1 = \prod |z_n - z_m^*| \, |z_m - z_n^*|, \quad P_2 = \prod |z_n - z_n^*|,$$

where $m > n$ and $m, n = 1, 2, \cdots, r$, he shows that $P_1 < k^{2r} r^r$ and $P_2 \leqq (4(k-1)/r)^r$ where $k = \Omega(f)$ and hence

$$1 \leqq |R(f, g)| = P_1 P_2 < (k^2(k-1))^r.$$

M. Marden (Milwaukee, Wis.).

Parodi, Maurice. Application d'un théorème de M. Lidskii à la recherche des limites supérieure et inférieure des parties réelles des zéros d'un polynome. C. R. Acad. Sci. Paris **233**, 1411–1412 (1951).

The real parts of the zeros of the polynomial

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

lie between the limits

$$-\tfrac{1}{2}[+a_1 + (a_1^2 + (1-a_2)^2 + \cdots + a_n^2)^{\frac{1}{2}}] - \cos \pi/n$$

and

$$\tfrac{1}{2}[-a_1 + (a_1^2 + (1-a_2)^2 + \cdots + a_n^2)^{\frac{1}{2}}] + \cos \pi/n.$$

E. Frank (Chicago, Ill.).

Varšavskiĭ, L. A. On conditions of stability of linear systems. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 907–919 (1951). (Russian)

A linear system is known to be stable if the zeros of the polynomial $\Delta(p)$ lie in the left half of the complex plane. In certain applications (the author considers only electrical systems) $\Delta(p) = A(p) + pRC(p)$ where $A(p)$ and $C(p)$ are even polynomials with real coefficients, and $A(p)$ is explicitly known. The problem is to determine the coefficients in $C(\sqrt{p}) = p^n + m_{n-1} p^{n-1} + \cdots + m_0$ so that the system is stable. It is known that the zeros of $A(p)$ and $C(p)$ are simple, lie on the imaginary axis, and separate each other. From this, the author deduces that a necessary condition for the stability of the system is that the point $(m_{n-1}, m_{n-2}, \cdots, m_0)$ in real Euclidean n-space, lies in a certain region bounded by the hyperplanes $\alpha_j^n + \sum_{k=0}^{n-1} m_k \alpha_j^k = 0$, $j = 1, 2, \cdots, n$, where the zeros of $A(p)$ are $\pm\sqrt{\alpha_j}$, $\alpha_j < 0$, $j = 1, 2, \cdots, n$. This suggests a study of the quotients of certain determinants, in which the author rediscovers some elementary results; see Muir [The theory of determinants . . . , vol. 1, 2nd ed., pp. 325–342 and vol. 3, pp. 144–148, Macmillan, London, 1906, 1920] and Goodman [Trans. Amer. Math. Soc. **63**, 175–192 (1948); these Rev. **9**, 421].

A. W. Goodman (Lexington, Ky.).

Lotkin, Mark. Polynomials having a root approaching π. Amer. Math. Monthly **59**, 96–98 (1952).

Block, H. D., and Thielman, H. P. Commutative polynomials. Quart. J. Math., Oxford Ser. (2) **2**, 241–243 (1951).

The authors prove that the only entire sets of commutative polynomials [i.e. such $p_n(x)$, $n = 1, 2, \cdots$, that

$p_m[p_n(x)]=p_n[p_m(x)]$ for every m, n] are the following:

$$P_n(x)=[(Ax+B)^n-B]/A,$$
$$T_n(x)=(1/A)\{\cos[n \arccos(Ax+B)]-B\},$$

$A\neq 0$, $n=1, 2, \cdots$ [in (II), p. 241 one bracket is misprinted]. Even if there exists one pair of polynomials $p_m(x)$, $p_n(x)$ such that $p_m[p_n(x)]=p_n[p_m(x)]$, then either $p_n(x)=P_n(x)$ or $p_n(x)=T_n(x)$ or there exists a number k and a function $f(x)$ for which $p_n(x)=(1/A)\ [f^k(Ax+B)-B]$, $[f^1(x)=f(x), f^k(x)=f(f^{k-1}(x))]$, or $p_m(x)$ is an odd function for which $p_m[-p_n(x)]=-p_n[p_m(x)]$. The authors derive, that

$$\sum_{k=1}^{n}\frac{a^{k-1}(n+k-1)!}{(n-k)!(2k-1)!k}\left(m\sum_{j=1}^{m}\frac{a^{j-1}(m+j-1)!(x-b)^j}{(m-j)!(2j-1)!j}\right)^k = m\sum_{k=1}^{mn}\frac{a^{k-1}(mn+k-1)!(x-b)^k}{(mn-k)!(2k-1)!k}.$$

J. Aczél (Miskolc).

du Plessis, N. A note about the derivatives of Legendre polynomials. Proc. Amer. Math. Soc. **2**, 950 (1951).

This is a short proof of the known relations [see, e.g., Grosswald, Proc. Amer. Math. Soc. **1**, 553–554 (1950); these Rev. **12**, 178] $P_n^{(r)}(1)=(n+r)!/2^r r!(n-r)!$ and $P_n^{(r)}(-1)=(-1)^{n+r}(n+r)!/2^r r!(n-r)!$, where $P_n(x)$ is the nth Legendre polynomial. *E. Grosswald.*

Special Functions

Michalup, Erich. Über die Stirling'sche Fakultätenformel. Statist. Vierteljschr. **2**, 117–119 (1949).

Durch die Eulersche Transformation wird die Stirling'sche Formel

$$x!=(2\pi x)^{\frac{1}{2}}e^{-x}x^x \exp\left\{\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)x^{2k-1}}\right\}$$

umgesetzt in

$$x!=(2\pi x)^{\frac{1}{2}}e^{-x}x^x \exp\left\{\frac{5x}{2(30x^2+1)}+\frac{265x}{14(30x^2+1)^2}+\cdots\right\}.$$

S. C. van Veen (Delft).

Tricomi, F. G. Asymptotische Eigenschaften der unvollständigen Gammafunktion. Math. Z. **53**, 136–148 (1950).

Der Verf. betrachtet die unvollständige Gammafunktion $\gamma(\alpha, x)=\int_0^x e^{-t}t^{\alpha-1}dt$, das Ashenbrödel der Funktionen.

$$\gamma(\alpha, x)=\Gamma(\alpha)x^\alpha\gamma^*(\alpha, x), \quad \text{mit} \quad \gamma^*(\alpha, x)=\sum_{n=0}^{\infty}x^n/\Gamma(\alpha+n+1);$$
$$\gamma_1(\alpha, x)=\Gamma(\alpha)x^\alpha\gamma^*(\alpha, -x); \quad \Gamma(\alpha, x)=\Gamma(\alpha)-\gamma(\alpha, x).$$

Die Bestimmung des asymptotischen Verhaltens dieser Funktionen, wenn $|\alpha|\to\infty$ bei beschränkt bleibendem $|x|$, bietet keine Schwierigkeit dar. Am meisten interessant ist zu untersuchen, was geschehen wird, wenn $|\alpha|$ und $|x|$ gleichseitig wachsen. In dieser Untersuchung sind zwei Fälle zu unterscheiden: 1) allgemeiner Fall: α und x nicht annäherend gleich; 2) spezieller Fall: x in der Nähe von $\alpha-1$. Im ersten Fall wird gefunden:

$$\Gamma(\alpha+1, x)\sim e^{-x}x^{\alpha+1}\sum_{m=0}^{\infty}\frac{F^{(m)}(0)}{(x-\alpha)^{m+1}},$$

wo $F(t)=e^{-\alpha t}(1+t)^\alpha$,

$$F^{(m)}(0)=(-1)^m m!\sum_{n=0}^{[m/2]}[(-\alpha)^n/n!]C_{m-2n}^{(n)}.$$

Die $C_k^{(n)}$ wird bestimmt durch

$$(\tfrac{1}{2}+\tfrac{1}{3}z+\tfrac{1}{4}z^2+\cdots)^n=\sum_{k=0}^{\infty}C_k^{(n)}z^k \quad (n=0, 1, \cdots).$$

Für die Gültigkeit der asymptotischen Entwicklung ist hinreichend, dass

$$|\alpha^{1/2}/(x-\alpha)|\to 0, \quad |\arg x^{1/2}/(x-\alpha)|<3\pi/4.$$

Beim zweiten Falle werden zwei Unterfälle unterschieden: 2a) Re $(\alpha)>0$:

$$\gamma(\alpha+1, \alpha+(2\alpha)^{1/2}y)$$
$$=\Gamma(\alpha+1)\{\tfrac{1}{2}\mp\mathrm{Erf}\,(\sqrt{y^2})-\tfrac{1}{3}(2/\alpha\pi)^{1/2}(1+y^2)e^{-y^2}+O(\alpha^{-1}),$$

wo das obere (untere) Vorzeichen gilt je nachdem

$$\tfrac{1}{2}\pi<|\arg y\alpha^{-1/2}|<\pi \qquad (|\arg y\alpha^{-1/2}|<\tfrac{1}{2}\pi)$$

ist. 2b) Re $(\alpha)<0$. Man findet speziell für α reell

$$\Gamma(\alpha)\gamma_1(1-\alpha, \alpha+(2\alpha)^{1/2}y)$$
$$=-\pi\cot\,(\alpha\pi)+2\pi^{1/2}\,\mathrm{Erfi}\,(y)+\tfrac{2}{3}(2\pi/\alpha)(1+y^2)e^{y^2}+O(\alpha^{-1}),$$

α, y reell, $\alpha>0$, $\alpha\to\infty$, $y=o(1)$. Die Arbeit schliesst mit einigen Bemerkungen über die reellen Nullstellen von $\gamma^*(\alpha, x)$ bei reellen α. Für $\alpha>0$ oder $\alpha=0, -1, -2, \cdots$: keine reelle Nullstelle. Für $\alpha<0$, $[-\alpha]$ gerade: eine negative Nullstelle; $[-\alpha]$ ungerade: eine negative und eine positive Nullstelle. Der Verf. gibt asymptotische Darstellungen für diesen Nullstellen. Zum Schluss gibt er ohne Beweis einige gut konvergente Reihenentwicklungen für die unvollständigen Gammafunktionen und ihren Sonderfall $\alpha=\frac{1}{2}$ (Fehlerintegral); z.B.:

$$\gamma(\alpha, x)=\Gamma(\alpha)e^{-x}x^{\alpha/2}\sum_{n=0}^{\infty}e_n(-1)x^{n/2}I_{n+\alpha}(2\sqrt{x}), \quad \alpha>\tfrac{1}{2},$$

wo $e_n(x)=\sum_{k=0}^{n}x^k/k!$. *S. C. van Veen* (Delft).

Radon, Brigitte. Sviluppi in serie degli integrali ellittici. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) **2**, 69–109 (1950).

In dieser Arbeit werden die Entwicklungen der vollständigen und unvollständigen elliptischen Integrale erster, zweiter und dritter Art von Legendre betrachtet.

Integrale erster Art:

$$F(\varphi, k)=\int_0^\varphi[1-k^2\sin^2\psi]^{-1/2}d\psi;$$

Integrale zweiter Art:

$$E(\varphi, k)=\int_0^\varphi[1-k^2\sin^2\psi]^{1/2}d\psi;$$

Integrale dritter Art:

$$\Pi(\varphi, \rho, k)=\int_0^\varphi[1+\rho\sin^2\psi]^{-1}[1-k^2\sin^2\psi]^{-1/2}d\psi;$$

$0\leqq\varphi\leqq\frac{1}{2}\pi$, $|k|<1$, k reell. Vollständig für $\varphi=\frac{1}{2}\pi$;

$$F(\tfrac{1}{2}\pi, k)=K(k), \quad E(\tfrac{1}{2}\pi, k)=E(k).$$

Diese Integrale werden entwickelt in Potenzreihen nach:

k^2, $k'^2=1-k^2$, $\xi=k^2+\rho$ (nur für die Integrale dritter Art);
$\left.\begin{array}{l} z=\Delta(\varphi, k)-\cos\varphi \\ u=1-\Delta(\varphi, k)\end{array}\right\}$ (nur für die unvollständigen Ingrale),

wo $\Delta(\varphi, k)=[1-k^2\sin^2\varphi]^{1/2}$. Diese Entwicklungen werden

hergeleitet aus partiellen Differentialgleichungen für die genannten Funktionen, z.B.

$$k(1-k^2)\frac{\partial^2 F}{\partial k^2}+(1-3k^2)\frac{\partial F}{\partial k}-kK=-\frac{k\sin\varphi\cos\varphi}{\Delta^3(\varphi,k)},$$

$$2\xi(\xi-1-\rho)\frac{\partial\Pi}{\partial\xi}+(\xi-1-\rho)\Pi=-E+(\xi-\rho)\frac{\sin\varphi\cos\varphi}{\Delta(\varphi,k)}.$$

Die Entwicklungen der Integrale dritter Art werden besonders betrachtet. Bei der Betrachtung der Konvergenzgebiete werden theoretische und praktische Konvergenz unterschieden. Eine Reihe wird praktisch konvergent genannt, wenn sie starker konvergiert als die Reihe $\sum_0^\infty 2^{-\nu}$. Die Endergebnisse sind in eine Liste zusammengefasst worden, die 9 Entwicklungen für die Integrale erster Art, 5 für die zweiter Art, und 9 für die dritter Art, nebst ihren Konvergenzgebieten enthält. *S. C. van Veen* (Delft).

Shabde, N. G. On some results involving Legendre functions. Gaṇita 1, 103–104 (1950).

This paper contains two results. The second, the expansion of a Legendre function of the second kind in series of Hermite polynomials, is known, and the first one, regarding the integral

$$\int_{-1}^{1} P_m(x)Q_n(x)dx$$

for unrestricted m and n is not correct except when m or $n-\frac{1}{2}$ is an integer. The author's criticism of a formula by Nielsen is unsound. *A. Erdélyi* (Pasadena, Calif.).

Nanjundiah, T. S. A note on an inequality of P. Turán for Legendre polynomials. Half-Yearly J. Mysore Univ. Sect. B., N.S. 11, 57–61 (1950).

Several authors discussed the inequality of Turán

$$\Delta_n(x)=[P_n(x)]^2-P_{n-1}(x)P_{n+1}(x)\geqq 0,\quad -1\leqq x\leqq 1,\ n\geqq 1.$$

A proof can be based on the identity

$$\Delta_n''(x)=-2[n(n+1)]^{-1}[P_n'(x)]^2$$

for which the author of the present note gives a derivation. Further, the inequality

$$D_n(x)=[P_n'(x)]^2-P'_{n-1}(x)P'_{n+1}(x)>0,\quad x \text{ arbitrary, real,}$$

is proved. [For $-1\leqq x\leqq 1$ more is shown in the paper of the reviewer [Bull. Amer. Math. Soc. 54, 401–405 (1948); these Rev. 9, 429] quoted by the author.] A simple proof is based on the remarkable identity

$$(1-x^2)D_n(x)=n(n+1)\Delta_n(x).$$

G. Szegö (Stanford University, Calif.).

Peiser, Alfred M. Uniform approximations to a class of Bessel functions. Proc. Amer. Math. Soc. 1, 650–661 (1950).

By a straight-forward application of Laplace transforms the solution of

$$\frac{\partial^2 t}{\partial x\partial y}+\frac{\partial t}{\partial x}+\frac{\partial t}{\partial y}=0$$

(when boundary values $t(x,0)$ and $t(0,y)$ are specified) may be obtained in the form

$$t(x,y)=t(x,0)e^{-y}+\int_0^x t(u,0)K_1(x-u,y)du$$
$$+\int_0^y\left[t(0,v)+\frac{\partial t(0,v)}{\partial v}\right]K_0(y-v,x)dv$$

where $K_n(x,y)=(y/x)^{n/2}e^{-x-y}I_n(2(xy)^{1/2})$, $n=0,1,2,\cdots$. It is desirable to obtain approximations to $K_0(x,y)$ and $K_1(x,y)$ which hold uniformly in $0\leqq x<\infty$. The author obtains the following inequalities for $x\geqq 0$

$$\left|K_0(x,y)-\frac{1}{(2y+1)^{1/2}}\phi\left(\frac{x-y-1}{(2y+1)^{1/2}}\right)\right|<\frac{0.321}{2y+1}$$
$$+\frac{5.101}{(2y+1)^{3/2}}+\frac{0.894}{(2y+1)^{1/2}}e^{-y/5},\quad y\geqq 10,$$

$$\left|K_1(x,y)-\frac{1}{(2y)^{1/2}}\phi\left(\frac{x-y}{(2y)^{1/2}}\right)\right|<\frac{0.167}{y}$$
$$+\frac{0.800}{y^{3/2}}+\frac{0.694}{y^{1/2}}\cdot e^{-y/5},\quad y\geqq 10,$$

where $\phi(x)$ is the normal probability function

$$\phi(x)=(2\pi)^{-1/2}e^{-x^2/2}.$$

Asymptotic expansions for the functions K are stated, e.g.:

$$K_1(y+x(2y)^{1/2},y)=\frac{1}{(2y)^{1/2}}\phi(x)-\frac{1}{4y}\phi^{(3)}(x)$$
$$+\frac{1}{4y(2y)^{1/2}}[\phi^{(4)}(x)+\tfrac{1}{9}\phi^{(6)}(x)]+O\left(\frac{1}{y^2}\right).$$

S. C. van Veen (Delft).

Bagchi, Hari Das, and Chatterji, Phatik Chand. Note on Weber's parabolic cylinder function $D_n(z)$ and its associated equations (functional and differential). Ann. Scuola Norm. Super. Pisa (3) 5, 71–80 (1951).

Weber's parabolic cylinder function $D_n(z)$ satisfies the two functional equations (I) $f_{n+1}(z)-zf_n(z)+nf_{n-1}(z)=0$, (II) $f_n'(z)+\frac{1}{2}zf_n(z)-nf_{n-1}(z)=0$ and the differential equation (A) $\omega''+(n+\frac{1}{2}-\frac{1}{4}z^2)\omega=0$. The authors point out that any two of these equations imply the third and make a few remarks on the common solutions. They do not give any new results on parabolic cylinder functions. *A. Erdélyi.*

Nørlund, N. E. Hypergeometric functions. Mat. Tidsskr. B. 1950, 18–21 (1950). (Danish)

Der Verf. betrachtet die lineare Differenzengleichung

$$(1)\quad \sum_{s=0}^{n}\binom{\beta+n}{s}\overset{s}{\Delta}Q(x)\overset{n-s}{\underset{-1}{\Delta}}u(x)=\sum_{s=0}^{n-1}\binom{\beta+n-1}{s}\overset{s}{\Delta}R(x)\overset{n-1-s}{\underset{-1}{\Delta}}u(x).$$

In ihr bedeutet $Q(x)$ ein Polynom in x vom $n$ten Grade, und $R(x)$ ein Polynom vom Grade $<n$. Gesetzt wird:

$$Q(x+n)=\prod_{k=1}^{n}(x+\alpha_k),$$

$$Q(x+n)-R(x+n-1)=\prod_{k=1}^{n}(x-\beta-\gamma_k),$$

$\alpha_k-\alpha_l,\ \beta_k-\beta_l\neq 0,\ \pm 1,\ \cdots\ (k\neq l)$. Die Substitution

$$u(x)=\int_0^1 z^{x-1}(1-z)^\beta y(z)dz$$

liefert Lösungen von (1) in hypergeometrischer Form, wenn eine Lösung der hypergeometrischen Differentialgleichung

$$\prod_{k=1}^{n}(\theta-\alpha_k)y-z\prod_{k=1}^{n-1}(\theta+\beta+\gamma_k)y=0\quad\left(\theta y=z\frac{dy}{dz}\right)$$

bedeutet. Mehrere Fundamentalsysteme von Lösungen von (1) werden angegeben. *S. C. van Veen* (Delft).

Mersman, W. A. A new form of solution of Hermite's equation. J. Math. Physics **29**, 191–197 (1950).

The classical methods of solution of a linear, homogeneous differential equation of the second order

$$(1)\qquad p(z)\frac{d^2u}{dz^2}+q(z)\frac{du}{dz}+r(z)u=0,$$

to obtain a second solution when one solution $y(z)$ is already known, lead to $u=yv$, with

$$v=\int\frac{e^Q}{y^2}dz,\quad Q=-\int\frac{q}{p}dz;$$

e.g., for Hermite's equation $v=\int e^{z^2}\{H_n(z)\}^{-2}dz$, where $H_n(z)$ is the Hermite polynomial of degree n. Unfortunately this integral representation of v is usually difficult to evaluate analytically or numerically. By the substitution $u=yv+w$ a second solution of (1) is obtained by the author by chosing v to be any solution of $pd^2v/dz^2+qdv/dz=0$ (a suitable choice is $v=\int e^Q dz$, $Q=-\int qp^{-1}dz$) and then a second solution is $u=yv+w$, provided that w is any solution of

$$p\frac{d^2w}{dz^2}+q\frac{dw}{dz}+rw=-2pe^Q\frac{dy}{dz}.$$

This method is especially powerful if the coefficients of the differential equation are polynomials and if the one solution already known is a polynomial. Examples: Hermite's equation of integral order $d^2u/dz^2-2zdu/dz+2nu=0$, $n=0, 1, 2, \cdots$,

$$u=AH_n(z)+B\left[H_n(z)\int_0^z e^{t^2}dt+G_n(z)e^{z^2}\right]$$

with

$$G_n(z)=\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^{k+1}\frac{(n-k-1)!}{(n-2k-1)!}\sum_{j=0}^{k}\frac{n!}{j!(n-j)!}(2z)^{n-2k-1}$$
$$(n=1, 2, \cdots),\quad G_0(z)=0.$$

In a boundary-layer problem solutions $u(z)$ of

$$d^2u/dz^2+2zdu/dz-4nu=0\quad(n=0, 1, 2, \cdots)$$

are required which are valid for $z>0$ and which vanish at $z=\infty$. In the way sketched above the solution is found to be

$$u(z)=A\left[H_{2n}(iz)2\pi^{-1/2}\int_z^\infty e^{-t^2}dt-2\pi^{-1/2}K_{2n}(z)e^{-z^2}\right]$$

with

$$K_{2n}=(-1)^n\sum_{k=0}^{n-1}\frac{(2n-k-1)!}{(2n-2k-1)!}\sum_{j=0}^{k}\frac{(2n)!}{j!(2n-j)!}(2z)^{2n-2k-1}$$
$$(n=1, 2, \cdots),\quad K_0=0.$$

In the same way Laguerre's equation of integral order $zd^2u/dz^2+(1-z)du/dz+nu=0$ is investigated.

S. C. van Veen (Delft).

Harmonic Functions, Potential Theory

Choquet, Gustave. Capacités. Premières définitions. C. R. Acad. Sci. Paris **234**, 35–37 (1952).

The author defines certain classes of non-additive set functions which he calls capacities. Briefly, suppose E is a topological space, $\mathcal{E}$ is a family of subsets of E, and f is a real-valued increasing set function defined on $\mathcal{E}$ which is "continuous on the right". Then f is called a capacity on $\mathcal{E}$. For any set $A\epsilon\mathcal{E}$, outer and inner capacities are defined with respect to f, and A is said to be capacitable if they coincide. If $A\epsilon\mathcal{E}$, or if A is open, then A is capacitable. More specialized definitions are given if $\mathcal{E}$ is additive (or multiplicative), i.e., if $\mathcal{E}$ contains the union (or intersection) of any two of its elements. If $\mathcal{E}$ consists of the closed subsets of E, a conjugate capacity $\bar{f}$ is defined on $\mathcal{E}$ by putting $\bar{f}(X)=-f(CX)$, where CX is the complement of X.

W. Rudin.

Rudin, Walter. Positive infinities of potentials. Proc. Amer. Math. Soc. **2**, 967–970 (1951).

Soit dans R^3 un ensemble fermé E de capacité nulle; en modifiant un peu la démonstration de G. C. Evans, l'auteur construit une distribution positive absolument continue (mais évidemment non portée par E) dont le potentiel est infini en tout point de E, fini ailleurs. Le rapporteur fait observer que ce résultat peut s'obtenir sans faire appel à la théorie du diamètre transfini.

J. Deny (Strasbourg).

Rudin, Walter. Green's second identity for generalized Laplacians. Proc. Amer. Math. Soc. **2**, 970–972 (1951).

Extension de l'identité classique de Green à des fonctions continues admettant un laplacien généralisé satisfaisant à des conditions précisées dans un article précédent de l'auteur [Trans. Amer. Math. Soc. **68**, 278–286 (1950); ces Rev. **11**, 663].

J. Deny (Strasbourg).

Minakshisundaram, S. Zeta functions on the unitary sphere. Canadian J. Math. **4**, 26–30 (1952).

In analogy to the case of the ordinary sphere with the ordinary metric, which the author has previously discussed [J. Indian Math. Soc. **13**, 41–48 (1949); these Rev. **11**, 108], he now discusses for the complex projective space with Fubini metric the behavior of (*) $\sum_1^\infty \varphi_n(P)\overline{\varphi_n(Q)}/\lambda_n^s$ where $\{\lambda_n, \varphi_n\}$ are eigenvalues and eigenfunctions of the Laplacean based on the metric; and he finds again that this is analytic everywhere in s except that for $P=\varphi$ there are simple poles at $s=1, 2, \cdots, k$. [Reviewer's comment. The author again raises the question of finding a "functional equation" for this "Zeta function", but the expression he finds for (*) will actually be such a functional equation if the integral be represented as an infinite sum of its residues. It is true that the latter infinite sum has now no immediate resemblance to the original series (*), as it has for the classical L-series, but such a resemblance is the exception rather than the rule, and must not be expected to be present if no underlying theta relation, with two sides similar, is known to be available to start with.]

S. Bochner (Princeton, N. J.).

Voelker, Dietrich. Singular solutions of the potential equation in the case of the Dirichlet and Neumann problems in the first quadrant. Revista Unión Mat. Argentina **15**, 32–37 (1951). (Spanish)

L'auteur utilise la transformation de Laplace à 2 variables pour trouver des familles de fonctions harmoniques dans le premier quadrant qui s'annulent ou dont les dérivés normales s'annulent sur les axes de coordonnées.

M. Brelot (Grenoble).

Birindelli, Carlo. Nuova trattazione di problemi al contorno di uno strato, per l'equazione di Poisson in tre variabili. II. Rivista Mat. Univ. Parma **2**, 235–263 (1951).

This is the second of a three part work on the solution of Poisson's equation, $\Delta u=f(x, y, z)$ in the slab $0<z<a$ by a

series expansion method. For a more complete statement of the problem and of the expansion method, see the review of part I [same Rivista 2, 77–102 (1951); these Rev. 13, 131] where questions of uniqueness and convergence of the expansion are treated. Part II is devoted to showing the continuity of the constructed solution function and its derivatives, and to proving that they satisfy the stated boundary conditions. The proofs are quite complicated, and involve detailed analysis of series of trigonometric and Bessel functions. *J. W. Green* (Princeton, N. J.).

Sugawara, Masao. On a system of differential equations. J. Math. Soc. Japan 3, 181–194 (1951).

The paper deals with potential problems for matrices whose elements are functions defined in a domain D with boundary Γ in euclidean space. Let H denote a constant symmetric positive definite matrix of order n. Then for any $m\times n$ matrices U, V the bilinear forms

$$[U,V]=\int_D UV'dv;\quad \{U,V\}=\int_\Gamma UHV'dw+\int_D\sum_i\frac{\partial U}{\partial x_i}\frac{\partial V'}{\partial x_i}dv$$

are introduced. The first proper function-matrix U_1 is defined as that matrix with $[U,U]=I$, for which the trace of $\{U,U\}$ has the least value. $U=U_1$ satisfies the conditions

$$\Delta U+LU=0\quad\text{in } D,\qquad dU/dn+UH=0\quad\text{on }\Gamma,$$

where L is the constant matrix given by $\{U_1,U_1\}$. The first proper "value" is defined as the matrix $K_1=L$. The higher proper function-matrices and proper values are defined by the same minimum property with the additional side condition to be orthogonal to the preceding ones in the $[U,V]$ sense. It is shown that the determination of the proper values for the matrix problem can be reduced completely to the corresponding problem for ordinary functions. Similarly the fact that any function-matrix F satisfying $dF/dn+FH=0$ on Γ can be expanded in a series of proper function matrices can be reduced to the corresponding fact for ordinary functions. *F. John* (New York, N. Y.).

Kalandiya, A. I. The solution of a fundamental boundary problem for the equation $\Delta^n u=0$ in a doubly connected region. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 17, 131–168 (1949). (Georgian. Russian summary)

The fundamental n-harmonic boundary value problem for a connected open set T of the $z=x+iy$ plane consists in the determination of a real-valued function $u(x,y)$ satisfying the polyharmonic equation

$$\Delta^n u=0,\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\quad n\geqq 1,$$

in T, and having prescribed normal derivatives of orders from zero to $n-1$ on the boundary L of T. In a companion paper [see the following review] the author considered the case when T is unbounded, and in a later paper [see the second following review] the case when T is bounded and multiply connected. In the present paper the case of bounded, doubly connected T is taken care of. Although the argument is still based on the results of N. I. Mushelišvili and I. N. Vekua mentioned in the reviews of the papers referred to above, the line of reasoning employed differs substantially from that in the last cited paper of the author. *J. B. Diaz* (College Park, Md.).

Kalandiya, A. I. The solution of a fundamental n-harmonic problem in the case of an infinite region. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 17, 169–189 (1949). (Russian. Georgian summary)

The "fundamental n-harmonic" boundary value problem for a connected open set T of the $z=x+iy$ plane consists in the determination of a real-valued function $u(x,y)$ satisfying the polyharmonic equation

$$\Delta^n u=0,\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\quad n\geqq 1,$$

in T, and having prescribed normal derivatives of orders from zero to $n-1$ on the boundary L of T. In the paper reviewed below the author considered this problem when T is a bounded, multiply connected set. The present paper deals with the case when T is an unbounded set whose boundary consists of a finite number of non-intersecting, sufficiently smooth, simple closed curves. The solution of the boundary value problem is sought among the class of solutions of the equation which satisfy the following growth condition at infinity:

$$\frac{\partial^{n-1}u}{\partial z^k\partial\bar z^{n-k-1}}=O(1),\quad k=0,1,\cdots,n-1.$$

The author shows the existence and uniqueness of the solution, basing his argument on the methods of N. I. Mushelišvili [C. R. (Doklady) Acad. Sci. URSS 2 (1934 I), 7–11; Math. Ann. 107, 282–312 (1932); Singular integral equations . . . , Moscow-Leningrad, 1946; these Rev. 8, 586], and I. N. Vekua [New methods for the solution of elliptic equations, Moscow-Leningrad, 1948; these Rev. 11, 598]. *J. B. Diaz* (College Park, Md.).

Kalandiya, A. I. The fundamental n-harmonic problem for multiply connected regions. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 185–198 (1951). (Russian)

Let T be a bounded, connected open set of the $z=x+iy$ plane, whose boundary $L=L_0+L_1+\cdots+L_p$ consists of $p+1$ non-intersecting simple closed curves, the curve L_0 encompassing the remaining ones. Assume that if L_j ($j=0,\cdots,p$) is given parametrically by $x=x_j(s)$, $y=y_j(s)$, where $0\leqq s\leqq l_j$, l_j the length of L_j, then the functions $x_j(s)$, $y_j(s)$ are of period l_j and have continuous derivatives of order $2n+2$, where $n\geqq 1$ is an integer. In the author's terminology, the "fundamental n-harmonic problem" consists in the determination of a real-valued function $u(x,y)$, defined on $T+L$, and such that

$$\Delta^n u=0,\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

in T, while on the boundary L:

$$u=f_0(s),\quad \frac{du}{d\nu}=f_1(s),\cdots,\quad \frac{d^{n-1}u}{d\nu^{n-1}}=f_{n-1}(s),$$

where ν denotes the outer normal of L. The functions $f_k(s)$ are given real-valued functions, having continuous derivatives, with respect to the arc length s, of order $\leqq 2n-k$ ($k=0,1,\cdots,n-1$); and $u(x,y)$ is required to be such that all its partial derivatives

$$\frac{\partial^{k+m}u}{\partial z^k\partial\bar z^m}\quad (k\leqq n,\ m\leqq n),$$

where

$$\frac{\partial^m}{\partial z^m}=\frac{1}{2^m}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^m,\quad \frac{\partial^m}{\partial\bar z^m}=\frac{1}{2^m}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^m,\quad m=0,1,\cdots,$$

are continuous on $T+L$. Boundary value problems for the same equation were considered by S. L. Sobolev [Mat. Sbornik N.S. **2(44)**, 465–499 (1937)] but the boundary conditions were taken "in the mean". The present author shows the existence and uniqueness of the fundamental n-harmonic problem, by showing that the solution of his problem is equivalent to the solution of a system of singular integral equations [N. I. Mushelišvili, Singular integral equations . . . , Moscow-Leningrad, 1946; these Rev. **8**, 586]. The argument is based on results contained in this book of Mushelišvili and in that of I. N. Vekua [New methods for the solution of elliptic equations, Moscow-Leningrad, 1948; these Rev. **11**, 598]. The n-harmonic problem for a simply connected domain had already been discussed by I. N. Vekua in his book, and the present paper constitutes an extension of his results to the multiply connected case.
J. B. Diaz (College Park, Md.).

Differential Equations

Gradštein, I. S. Differential equations in which various powers of a small parameter appear as coefficients of the derivatives. Doklady Akad. Nauk SSSR (N.S.) **82**, 5–8 (1952). (Russian)

Consider the systems

$$(1)\quad \begin{cases} \dfrac{dX}{dt}=f(X,Y,Z,t), \\[2mm] \eta\dfrac{dY}{dt}=h(X,Y,Z,t), \\[2mm] \eta^{1+\alpha}\dfrac{dZ}{dt}=g(X,Y,Z,t), \quad \alpha>0, \end{cases}$$

and

$$(2)\quad \frac{dx}{dt}=f(x,y,z,t),\quad h(x,y,z,t)=0,\quad g(x,y,z,t)=0,$$

where X, x, f are m-vectors, Y, y, h are μ-vectors and Z, z, g are ν-vectors. The author compares the solutions of the two systems and states (without proof) a very complicated theorem under which, for a limited time, and when they have the same initial values, the solution of (1) tends towards the solution of (2). The paper generalizes the result in a preceding paper [same Doklady **81**, 985–986 (1951); these Rev. **13**, 460].
S. Lefschetz (Princeton, N. J.).

Basov, V. P. On solutions of a class of systems of linear differential equations. Doklady Akad. Nauk SSSR (N.S.) **80**, 301–304 (1951). (Russian)

The author considers the linear system:

$$dx_s/dt=t^{-\alpha}\sum_{\sigma=1}^{m}r_{s\sigma}x_\sigma+t^{\beta}\sum_{\sigma=m+1}^{n}r_{s\sigma}x_\sigma \quad (s=1,\cdots,m),$$

$$dx_s/dt=t^{-\alpha}\sum_{\sigma=1}^{m}r_{s\sigma}x_\sigma+t^{\beta}\sum_{\sigma=m+1}^{n}(p_{s\sigma}+q_{s\sigma})x_\sigma \quad (s=m+1,\cdots,n).$$

It is assumed that the $r_{s\sigma}$ and $q_{s\sigma}$ are in general complex-valued functions of t which are bounded for $t\geqq T>1$, and $q_{s\sigma}\to0$, as $t\to\infty$. The $p_{s\sigma}$ are constants, and the matrix with these constants as elements has no characteristic roots with zero real part. The following theorem is stated: If $\alpha>1$, $\beta>-1$, then this system has solutions of the form

$$x_s=\delta_s^{(i)}+t^{1-\alpha}u_s^{(i)} \quad (s=1,\cdots,m),$$
$$x_s=t^{-(\alpha+\beta)}u_s^{(i)} \quad (s=m+1,\cdots,n;\ i=1,\cdots,m),$$

where the $u_s^{(i)}$ ($i=1,\cdots,m$; $s=1,\cdots,n$) are functions of t which are bounded for $t\geqq T$, and $\delta_s^{(i)}$ as 1 or 0 according as $s=i$ or $s\neq i$. (Presumably for each i, such a solution $x_s=x_s^{(i)}$, $s=1,\cdots,n$, exists.) This result is used to investigate the solutions of the system

$$dx_s/dt=\sum_{\sigma=1}^{n}(p_{s\sigma}+t^{-\gamma}q_{s\sigma})x_\sigma \quad (s=1,\cdots,n),$$

where γ and $p_{s\sigma}$ are real constants, $\gamma>0$, and $q_{s\sigma}$ are real continuous functions of t, bounded for $t\geqq T$.
E. A. Coddington (Cambridge, Mass.).

Donskaya, L. I. On the structure of the solutions of a system of three linear differential equations in the neighborhood of the irregular singular point $t=\infty$. Doklady Akad. Nauk SSSR (N.S.) **80**, 321–324 (1951). (Russian)

In slightly different notation, the homogeneous system $x'=Px$ ($'=d/dt$) is investigated, where $P=\sum_{m=0}^{\infty}P^{(m)}t^{-m}$ is a three-by-three matrix, t and $P^{(m)}$ are real, and $P^{(0)}$ is in canonical form. The detailed description of solution matrices of $x'=Px$ near $t=\infty$ is given in terms of certain exponentials, powers of t, and uniformly convergent series.
E. A. Coddington (Cambridge, Mass.).

Kneser, Hellmuth. Die Reihenentwicklung bei schwach singulären Stellen linearer Differentialgleichungen. Arch. Math. **2** (1949–1950), 413–419 (1951).

Let y be a vector considered as a matrix of one column and n rows, and suppose A is an n by n matrix of functions defined in the vicinity of a point $x=a$. The point $x=a$ is said to be a weak singular point for the homogeneous system $y'=Ay$ ($'=d/dx$) if A has at most a simple pole at $x=a$. The author shows how the principal theorem concerning the representation of the solutions of $y'=Ay$ can be obtained without using the complete Jordan canonical form of a matrix. A convergence proof is given for formal solutions obtained. (Without using the elementary divisor theory, Wintner [Amer. J. Math. **68**, 185–213 (1946); these Rev. **8**, 71] gave a corresponding result for the solution matrices of $y'=Ay$.)
E. A. Coddington (Cambridge, Mass.).

Putnam, Calvin R., and Wintner, Aurel. Linear differential equations with almost periodic or Laplace transform coefficients. Amer. J. Math. **73**, 792–806 (1951).

The nature of the solutions of the equation

$$\sum_{h=0}^{n}f_h(t)d^hx/dt^h=0$$

is investigated, where $f_n(t)\equiv1$, and the other f_h are uniformly almost periodic functions on $-\infty<t<+\infty$ whose Fourier frequencies have a positive lower bound. It is shown that this equation has a non-trivial solution which is bounded, as $t\to\pm\infty$, and this is, up to a constant factor, the only bounded solution. Further, this solution is uniformly almost periodic, and its non-zero frequencies are linear combinations, with positive integral coefficients, of those for the f_h. The structure of n linearly independent solutions is given. Corresponding results hold for the case where the f_h are absolutely convergent Laplace integrals.
E. A. Coddington (Cambridge, Mass.).

Péyovitch, T. Sur les solutions asymptotiques des équations différentielles. Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 121–145. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. French summary)

The paper presents essentially a recapitulation of an earlier paper by the same author [Publ. Math. Univ. Belgrade 1, 12–54 (1932)]. By the methods of power series expansions and of successive approximations he obtains asymptotic solutions of linear differential equations with coefficients that converge as the independent variable increases indefinitely. *M. Golomb* (Lafayette, Ind.).

***Boerdijk, Arie Hendrik. Vector representations of differential equations, their solutions and the derivatives thereof.** Thesis, Technische Hogeschool te Delft. Uitgeverij Waltman, Delft, 1951. 179 pp.

Exponentially damped sinusoidal oscillations are in the usual way graphically represented as projections on a fixed line through the origin of a uniformly rotating vector of diminishing length. The derivatives can in various ways be similarly represented in the same diagram. The author shows how this representation can be used to advantage for the illustration and mathematical analysis of vibration problems that can be strictly or approximately described in terms of such damped sinusoidal oscillations. In addition to familiar phenomena occurring in the theory of second order linear differential equations with constant coefficients and sinusoidal or piecewise constant forcing terms, the author investigates some non-linear differential equations which are amenable to this approach, because the non-linear coefficients vary very slowly, or because they are piecewise constant. Several cases of subharmonic resonance are studied, and the results are shown to agree with the experimental data. *W. Wasow* (Los Angeles, Calif.).

Golomb, Michael, and Usdin, Eugene. A theory of multidimensional servo systems. J. Franklin Inst. **253**, 29–57 (1952).

An n-dimensional linear servo system is a combination of n simple linear servomechanisms, with input signals $x_1(t), \cdots, x_n(t)$ and output signals $y_1(t), \cdots, y_n(t)$, having coupled error measuring devices. Explicitly, the typical output signal $y_k(t)$ is controlled by a linear combination of the n error signals $\epsilon_i(t)=x_i(t)-y_i(t)$. The use of matrix methods leads to a formulation of the theory of such systems which is formally similar to the theory of simple servo systems. The authors develop the fundamentals of this theory, with special emphasis on the estimation of the various instantaneous and average errors to which the output signals are subject. It is known that a proper choice of the controlling linear combinations of the $\epsilon_i(t)$, possibly together with a proper choice of certain other parameters, may result in an n-dimensional servo system being stable, even when some of the component simple servos are unstable. The authors discuss this possibility extensively for several typical cases, obtaining theoretical designs which make the systems stable and also, in a certain sense, optimum. *L. A. MacColl.*

Morduchow, M., and Galowin, L. On double-pulse stability criteria with damping. Quart. Appl. Math. **10**, 17–23 (1952).

The authors treat the effect of a double-pulse system with damping and especially the stability conditions adhering to such a system. By the use of known results, this problem can be solved simply and the authors formulate their results in the form of 4 conclusions. *M. J. O. Strutt* (Zurich).

Bogatyrev, O. M. Determination of the constants of integration in the solution of a differential equation of high degree. Èlektričestvo **1951**, no. 8, 74–81 (1951). (Russian)

For the case of a linear homogeneous differential equation with constant coefficients, algebraic formulae are given relating the constant of integration to the initial values. *N. Levinson* (Cambridge, Mass.).

Gel'fand, I. M., and Levitan, B. M. On the determination of a differential equation from its spectral function. Izvestiya Akad. Nauk SSSR. Ser. Mat. **15**, 309–360 (1951). (Russian)

This is a detailed account of the results sketched in an article of the same title [Doklady Akad. Nauk SSSR (N.S.) **77**, 557–560 (1951); these Rev. **13**, 240]. In the notation of the earlier review let the monotone function $\rho(\lambda)=2(\lambda/\pi)^{1/2}+\sigma(\lambda)$ for $\lambda\geqq 0$ and $\rho=\sigma(\lambda)$ for $\lambda<0$. Let $\int_{-\infty}^{0}\exp(|\lambda|^{1/2}x)d\rho(\lambda)$ exist for all $x>0$ and let $a(x)=\int_1^\infty \lambda^{-1}\cos(\lambda)^{1/2}x\,d\sigma(\lambda)$ be of class C_4. Then there exists a continuous function $q(x)$ and a constant h such that $\rho(\lambda)$ is the spectral function for $y''+(\lambda-q(x))y=0$, $0\leqq x<\infty$, $y(0,\lambda)=1$, $y'(0,\lambda)=h$. The relationship (*) $\phi(x,\lambda)=\cos(\lambda^{1/2}x)+\int_0^x K(x,t)\cos(\lambda^{1/2}t)dt$ plays a fundamental role. The problem on the finite interval is also considered. [Remark: The proof that $y=\phi(x,\lambda)$ defined by (*) satisfies the differential equation can be shortened considerably by proceeding as follows. The function $K(x,y)$ in (*) is determined by

$$I=K(x,y)+f(x,y)+\int_0^x K(x,t)f(t,y)dt=0$$

where $f(x,y)=\alpha(x+y)+\alpha(x-y)$ for some $\alpha(x)$ of class C_2. Take the partial derivatives I_{xx} and I_{yy} and reformulate I_{yy} by use of integration by parts. Consider now $I_{xx}-I_{yy}-q(x)I=0$. This turns out to be

$$J(x,y)+\int_0^x J(x,t)f(t,y)dt+[\tfrac{1}{2}dK(x,x)/dx-q(x)]f(x,y)=0$$

where $J(x,y)=K_{xx}-K_{yy}-q(x)K$ and where account has been taken of the fact that $K_y(x,0)=0$. If $q(x)$ is taken as $\tfrac{1}{2}dK(x,x)/dx$ then the integral equation for J becomes homogeneous and therefore has only the null solution. Thus $J=0$ and K satisfies $K_{xx}-K_{yy}-q(x)K=0$. Direct calculation now shows $y=\phi(x,\lambda)$ satisfies the differential equation (and also appropriate initial conditions).] *N. Levinson.*

Rapoport, I. M. On singular boundary problems for ordinary linear differential equations. Doklady Akad. Nauk SSSR (N.S.) **79**, 21–24 (1951). (Russian)

The paper concerns a fourth order ordinary formally self-adjoint differential equation $(ay'')''+(by')'+cy+\lambda y=0$ $('=d/dx)$ on the interval $0\leqq x<\infty$. Under certain assumptions on the coefficients a, b, c, and assuming a set of two initial conditions at $x=0$, the author shows how the generalized Parseval equality can be obtained for this equation. The principal method is the detailed investigation of the asymptotic behavior of the solutions of this equation. A rather complicated set of conditions is given which guarantees that the spectrum of this problem is the entire λ-axis (continuous spectrum), and the negation of one of these conditions is sufficient for the spectrum to be discrete. (There seem to be a few misprints. For example the expression in formula (1) should be equated to zero, and the second

boundary condition in (2) should have b replaced by 0. Reference should be made to the very general paper by Kodaira on this subject [Amer. J. Math. 72, 502–544 (1950) these Rev. 12, 103].) *E. A. Coddington.*

Dramba, C. Sur les multiplicités singulières des systèmes différentielles. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 1, 162–168 (1950). (Romanian. Russian and French summaries)

The author considers differential equations

$$dx_i = A_i(x_1, \cdots, x_n)dt \quad (i=1, \cdots, n).$$

It is assumed that the functional determinant of the functions A_i has rank $n-k$ and that all functions A_i vanish at the origin; there is then in general a k-dimensional variety M of singular points through the origin. It is shown how the solutions can be expressed in terms of k constants, which locate position on M and $n-k$ constants, which describe integral surfaces through the singular points.

W. Kaplan (Ann Arbor, Mich.).

★Stanković, B. Gewinnung von Differentialinvarianten durch Differenzierung. Premier Congrès des Mathématiciens et Physiciens de la R.P.F.Y., 1949. Vol. II, Communications et Exposés Scientifiques, pp. 33–36. Naučna Knjiga, Belgrade, 1951. (Serbo-Croatian. German summary)

The author observes that if there are r $(r<2n)$ functions $\varphi_1, \cdots, \varphi_r$ of the variables $x_1, \cdots, x_n, p_1, \cdots, p_n$ which form a functional group and are such that

$$F(x_1, \cdots, x_n, p_1, \cdots, p_n) = \Phi(\varphi_1, \cdots, \varphi_r)$$

then the differential equation $F=0$ $(p_k = \partial z/\partial x_k)$ has the differential invariants $\varphi_1, \cdots, \varphi_r$. *M. Golomb.*

Germay, R. H. Sur une extension d'un théorème de Jacobi pour l'intégration des équations aux dérivées partielles du premier ordre de forme résolue par rapport à la dérivée en x_1. Ann. Soc. Sci. Bruxelles. Ser. I. 65, 103–108 (1951).

In the holomorphic domain the equation

$$p_1 = f(x_1, \cdots, x_n; z, p_2, \cdots, p_n)$$

has k solutions $z_1, \cdots, z_k$ defined implicitly by

$$W_j(x_1, \cdots, x_n; z_1, \cdots, z_k) = 0$$

and satisfying for $x_1 = 0$ arbitrary equations

$$V_j(x_2, \cdots, x_n; z_1, \cdots, z_k) = 0$$

with Jacobian different from zero. The W's satisfy a linear homogeneous partial differential equation of the first order.

J. M. Thomas (Durham, N. C.).

Sauer, Robert. Projektive Beziehungen in der Charakteristikentheorie der partiellen Differentialgleichungen. Arch. Math. 2 (1949–1950), 420–423 (1951).

The Legendre transformation applied to a differential equation $af_{xx}+2bf_{xy}+cf_{yy}=0$ replaces the variables x, y, f by $\xi = f_x$, $\eta = f_y$, $\varphi = x\xi + y\eta - f$. It is shown that a linear homogeneous transformation applied to the 3 quantities $\xi, \eta, -\varphi$ corresponds to a projective transformation of the net of characteristics of the equation in the (x, y)-plane, which has the same matrix, when written in line coordinates. This fact yields a new proof of the theorem that the infinitesimal isometric deformations of a surface can be determined by quadratures, if those of a projectively related surface are known. *F. John* (New York, N. Y.).

Fourès-Bruhat, Yvonne. Théorème d'existence pour des systèmes d'équations aux dérivées partielles à quatre variables. C. R. Acad. Sci. Paris 234, 500–502 (1952).

The result reviewed previously [same C. R. 231, 318–320 (1950); these Rev. 12, 185] is extended to the case in which the coefficients of the second derivatives involve the first derivatives. *J. M. Thomas* (Durham, N. C.).

Fourès-Bruhat, Yvonne. Solution du problème de Cauchy pour des systèmes d'équations hyperboliques du second ordre non linéaires. C. R. Acad. Sci. Paris 234, 585–587 (1952).

The result of the paper reviewed above is now extended to n independent variables. *J. M. Thomas.*

Berezin, I. S. On Cauchy's problem for linear equations of the second order with initial conditions on a parabolic line. Amer. Math. Soc. Translation no. 62, 26 pp. (1952).

Translated from Mat. Sbornik N.S. 24(66), 301–320 (1949); these Rev. 11, 112.

Oleĭnik, O. A. On the second boundary problem for an equation of elliptic type with a small parameter in the highest derivatives. Doklady Akad. Nauk SSSR (N.S.) 79, 735–737 (1951). (Russian)

The partial differential equation

$$L_\epsilon(U) = \epsilon(U_{xx}+U_{yy}) + A(x, y)U_x + B(x, y)U_y + C(x, y)U = f(x, y)$$

is considered in a domain G with the boundary value for $\partial U/\partial n$ assigned on S, the boundary of G. It is assumed that $A^2+B^2>0$ in G and that $C<0$. The behaviour of $U(x, y, \epsilon)$ is compared with that of a solution of the degenerate first order equation obtained by putting $\epsilon=0$ in $L_\epsilon(U)=0$. Results are stated without proof. If s is arc length on S then the arcs S on which $B(x, y)dx/ds - A(x, y)dy/ds > 0$ play a special role much as in the results of the reviewer [Ann. of Math. (2) 51, 428–445 (1950); these Rev. 11, 439] for the Dirichlet problem for $L_\epsilon(U)=0$. *N. Levinson.*

Il'in, V. A. On the convergence of bilinear series of eigenfunctions. Uspehi Matem. Nauk (N.S.) 5, no. 4(38), 135–138 (1950). (Russian)

Let $\{u_i(p)\}$ and (λ_i) be the sequences of eigen-functions and eigen-values of the equation

$$\nabla^2 u + \lambda u = 0$$

in n dimensions, satisfying a set of homogeneous boundary conditions. In this preliminary abstract the author states the following results without detailed proof. (i) For the n-dimensional rectangular parallelepiped, the series

$$\sum_i \frac{u_i(P)u_i(Q)}{\lambda_i^{(n/2)}} \tag{1}$$

is not absolutely convergent (contrary to a statement by Courant and Hilbert [Methoden der mathematischen Physik, vol. 1, 2d. ed., p. 335, Springer, Berlin, 1931] for the case $n=2$), but the series

$$\sum_i \frac{u_i(P)u_i(Q)}{\lambda_i^{\frac{1}{2}(n-1)+\epsilon}} \tag{2}$$

is (possibly conditionally) convergent for $\epsilon>0$. (ii) For an arbitrary n-dimensional domain the series

$$\sum_i \frac{u_i(P)u_i(Q)}{\lambda_i^{(n/2)+\epsilon}} \tag{3}$$

is absolutely convergent for $\epsilon>0$, and the series

$$(4)\qquad \sum_i \frac{u_i(P)u_i(Q)}{\lambda_i^{(n/4)+\epsilon}}$$

is convergent in mean square for $\epsilon>0$. (iii) The conditional convergence of (2) is also asserted for certain regions more general than a rectangular parallelepiped. *F. Smithies.*

Yosida, Kôsaku. Integrability of the backward diffusion equation in a compact Riemannian space. Nagoya Math. J. 3, 1–4 (1951).

Let R be an orientable, compact Riemann space with metric $ds^2=g_{ij}(x)dx^idx^j$. The corresponding Fokker-Planck, or forward, diffusion equation is of the form

$$(1)\qquad \frac{\partial f}{\partial t}=Af(t,x)$$

where the operator A is given by

$$Af=g^{-1/2}(x)\frac{\partial^2}{\partial x^i\partial x^j}\{g^{1/2}(x)b^{ij}(x)f\}+g^{-1/2}(x)\frac{\partial}{\partial x^i}\{-g^{1/2}(x)a^i(x)f\}.$$

The backward diffusion equation is obtained on replacing A by its formal adjoint

$$A'f=b^{ij}(x)\frac{\partial^2 f}{\partial x^i\partial x^j}+a^i(x)\frac{\partial f}{\partial x^i}.$$

Previously [Ark. Mat. 1, 71–75 (1949); these Rev. 11, 443] the author has shown that (1) is integrable and that the solutions have the properties postulated by the probabilistic meaning of (1). Here an analogous result is obtained for the backward equation assuming the dimension of R is at least 2. The method depends on the consideration of the resolvent of the semigroup generated by A'. The underlying Banach space is, of course, that of continuous functions with the usual norm. *W. Feller* (Princeton, N. J.).

Yosida, Kôsaku. Integration of Fokker-Planck's equation with a boundary condition. J. Math. Soc. Japan 3, 69–73 (1951).

With the notations of the preceding review let G be a connected region of R and Γ its boundary. Assume the coefficients g_{ij}, b^{ij}, a^i to be infinitely differentiable. Applying Green's formula, the integral of $h(Af)-f(A'h)$ extended over G is transformed into an integral I extended over Γ. Denote the coefficient of h in the integrand by Ωf (so that Ω is a differential operator of first order). Consider the forward equation (1) in the space of integrable functions with the usual L-norm. If the dimension of R is at least 2 it is shown that (1) has a unique solution if and only if for no $m>0$ there exists a bounded solution h of $A'h=mh$ which satisfies the boundary condition which makes the integral I vanish identically. It can easily be shown that this condition is satisfied if $G=R$. Other applications are to be published.
W. Feller (Princeton, N. J.).

Squire, William. A problem in heat conduction. J. Appl. Phys. 22, 1508–1509 (1951).

In a medium of infinite extent moving with constant velocity there is a fixed point-source of heat that generates heat at a rate proportional to sin ωt or cos ωt. Using classical methods and the Dirac delta function, the author obtains a formula for the simple periodic temperatures with frequency ω in the medium. *R. V. Churchill.*

Žautykov, O. Cauchy's problem for a denumerable system of partial differential equations. Izvestiya Akad. Nauk Kazah. SSR. 60, Ser. Mat. Meh. 3, 85–90 (1949). (Russian. Kazak summary)

The Cauchy problem in question consists in the determination of a countable sequence of unknown functions: $z_1(x,y)$, $z_2(x,y)$, $\cdots$, $z_n(x,y)$, $\cdots$, satisfying the system of partial differential equations

$$\frac{\partial^2 z_k}{\partial x\partial y}=f_k\left(x,y,z_1,z_2,\cdots,\frac{\partial z_1}{\partial x},\frac{\partial z_2}{\partial x},\cdots,\frac{\partial z_1}{\partial y},\frac{\partial z_2}{\partial y},\cdots\right),\qquad k=1,2,\cdots,$$

and subject to the following Cauchy conditions on the line $x=0$:

$$z_k(0,y)=\varphi_k(y),\quad \frac{\partial z_k}{\partial x}(0,y)=\psi_k(y),\quad k=1,2,\cdots,$$

where the f_k, φ_k and ψ_k are given functions. It is shown, by the method of successive approximations, that the problem has one and only one solution, under suitable hypotheses on the given functions involved. For example, the functions

$$f_k(x,y,u_1,u_2,\cdots,p_1,p_2,\cdots,q_1,q_2,\cdots),\quad k=1,2,\cdots,$$

are taken to be "equicontinuous" and equibounded in all their variables, and to satisfy a certain Cauchy-Lipschitz condition with respect to the arguments $u_1,u_2,\cdots,p_1,p_2,\cdots,q_1,q_2,\cdots$. *J. B. Diaz* (College Park, Md.).

Teixidor, J. Solution in finite terms of the Cauchy problem for a family of partial differential equations of the 4th order. Collectanea Math. 3, 1–71 (1950). (Spanish)

Depuis environ une vingtaine d'années, comme application d'une suite de mémoires sur fonctionnels analytiques, L. Fantappié a donnée une représentation de la solution du problème de Cauchy-Kowalewsky pour une équation (ou un système d'équations) aux dérivées partielles linéaires à coefficients constants, qui se forme au moyen d'une succession d'intégrations dans le sens ordinaire et de déterminations de résidus par integration complexe sur un contour. C'est cette représentation que Fantappié nomme de "quadrature" ou "en termes finis". Le présent mémoire a pour but d'appliquer le procédé au cas particulier de l'équation du 4ème ordre

$$Q(\partial/\partial x,\partial/\partial y,\partial/\partial t)=f(t,x,y)$$

où Q est un polynôme homogène; et plus précisément on suppose que la courbe plane $Q(\xi,\eta,\zeta)=0$ soit le colimaçon de Pascal. Le travail comprend un premier chapitre destiné à l'étude géométrico-algébrique de cette courbe et trois autres chapitres qui, à moins de certains détails algébriques à l'égard de la séparation des branches de la courbe $Q=0$, suivent fidèlement le procédé indiqué par Fantappié. Dans l'opinion du référent c'est peut-être cette partie algébrique le lieu plus personnel, mais aussi de lecture moins claire et plus difficile du travail; principalement par-ce-qu'elle entre dans des détails que Fantappié semble avoir considérés comme superflus, y fort à raison dans son point de vue restreint au champ des fonctions analytiques, et qui pourraient au contraire avoir d'importance en cas d'étudier l'équation différentielle au point de vue des fonctions réelles.
B. Levi (Rosario).

Integral Equations

Sugawara, Masao. On the theory of linear integral equations with a symmetric kernel in the matrix-space. J. Fac. Sci. Univ. Tokyo. Sect. I. 6, 227–246 (1951).

The paper is an attempt to carry through the Hilbert-Schmidt theory of the linear integral equation with symmetric kernel in the form $\lambda(\varphi(s)-f(s))=\int\varphi(t)K(s,t)dt$ for the case where λ is a non-singular $m\times m$ constant matrix, $\varphi(s)$ and $f(s)$ are $m\times n$ matrices of functions, $K(s,t)$ is an $n\times n$ matrix of functions satisfying the condition $K(s,t)=\bar{K}(t,s)$, where $\bar{K}$ is the transpose of the matrix K, and multiplication is matrix multiplication. A characteristic constant matrix Λ and characteristic function $m\times n$ matrix $U(s)$ are determined by applying the calculus of variations to the problem of finding a matrix U such that $\int U\bar{U}=E$ (identity matrix) which minimizes the trace of

$$\iint U(s)K(s,t)\bar{U}(t)dsdt$$

and using the Lagrange multiplier method. A Schmidt orthogonalisation process for a finite number of matrix functions is possible. Unfortunately, the proofs of many results are invalid because the author forgets that $\int U\bar{U}$ is not a real number but a matrix. This is true of such things as a Bessel's inequality, and expansion theorems generalizing those of functions of the form $\int H(t)K(s,t)dt$ in terms of the characteristic functions and values of K.

T. H. Hildebrandt (Ann Arbor, Mich.).

Sibirani, Filippo. Soluzioni polinomiali di un tipo di equazioni integrali. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 6, 173–177 (1950).

The author investigates the Volterra integral equation

$$f(x)=\varphi(x)-\int_0^x K(x,y)\varphi(y)dy,$$

where $f(x)$ is a polynomial in x, and $K(x,y)$ is a homogeneous polynomial in x and y. He finds conditions for the solution $\varphi(x)$ to be also a polynomial; the conditions are too complicated to be given in detail. *F. Smithies.*

Gross, B. On the inversion of the Volterra integral equation. Quart. Appl. Math. 10, 74–76 (1952).

Let the function $k(x)$ be the Laplace transform of a known function $r(t)$, $k(x)=\int_0^\infty e^{-xt}r(t)dt$. The author derives an explicit formula that gives the unknown function $m(x)$ in the integral equation $k(x)=m(x)+\int_0^x k(x-t)m(t)dt$ in terms of integrals involving $r(t)$. Among additional conditions the method seems to require that the kernel $k(x)$ itself has a Laplace transform. *R. V. Churchill* (Ann Arbor, Mich.).

Nomokonov, M. K. On the simpleness of the second characteristic value of correlation integral equations. Doklady Akad. Nauk SSSR (N.S.) 72, 1021–1024 (1950). (Russian)

The author considers the integral equation

$$\text{(1)}\qquad \varphi(x)=\lambda\int_a^b \frac{F(x,y)}{p(x)}\varphi(y)dy,$$

where $F(x,y)>0$, $F(x,y)=F(y,x)$, $p(x)=\int_a^b F(x,y)dy$,

$$\int_a^b\int_a^b F(x,y)dxdy=1,\qquad \int_a^b\int_a^b \frac{F^2(x,y)}{p(x)p(y)}dxdy<\infty.$$

He shows that if

$$\Psi(x,y)=\int_a^y \frac{\partial}{\partial x}\frac{F(x,t)}{p(x)}dt$$

is of constant sign in $a\leqq x\leqq b$, $a\leqq y\leqq b$, then the second characteristic value of (1) is simple, and of sign opposite to that of $\Psi(x,y)$, and that the corresponding characteristic function is monotone. A corresponding result is proved for the second singular value when $F(x,y)$ is not symmetric.

F. Smithies (Cambridge, England).

Hornback, Joseph Hope. Integral equations related to the representation of functions by potentials. Abstract of a Thesis, University of Illinois, Urbana, Ill., 1952. 2+i pp.

Functional Analysis

Grothendieck, Alexandre. Quelques résultats sur les espaces vectoriels topologiques. C. R. Acad. Sci. Paris 233, 839–841 (1951).

Let K be a compact space, let $C(K)$ be the space of all continuous functions on K, and let E be a complete locally convex space. The author shows for a linear continuous operator U on $C(K)$ to E, that the following are equivalent: a) U is weakly compact, b) U transforms weakly convergent sequences into strongly convergent sequences, c) U transforms weak Cauchy sequences into weakly convergent sequences. Moreover, such a U transforms weakly compact subsets into compact subsets. He obtains as a corollary the fact that every linear continuous transformation of $C(K)$ into an L^1 space is weakly compact. The author then considers some integration problems. Suppose now that K is locally compact with a measure μ, and let E be a Banach space. Suppose further that $f(t)$ is a weakly measurable function on K to E. If f is locally weakly conditionally compact, then f is (what the author calls) weakly locally almost everywhere equal to a strongly measurable function.

R. S. Phillips (Los Angeles, Calif.).

Hukuhara, Masuo. Sur l'existence des points invariants d'une transformation dans l'espace fonctionnel. Jap. J. Math. 20, 1–4 (1950).

Let K be a convex subset of a locally convex linear space R and f a continuous self-mapping of K. The author proves that if $f(K)$ is contained in a compact subset of K, then f has a fixed point in K. This is a refinement of a well-known theorem of Tychonoff [Math. Ann. 111, 767–776 (1935)] stating that, if K itself is compact, f has a fixed point in K. The author's theorem may be obtained directly from the latter theorem if R is complete with respect to its natural uniformity. In the final section of the paper, the author outlines without detailed proof an extension of the definition and properties of the Leray-Schauder degree to mappings of an open set D of R into R, of the form $I-f$, where f is continuous and $f(D)$ is contained in a compact subset of R.

F. Browder (Boston, Mass.).

Vaccaro, Michelangelo. Sui funzionali analitici lineari definiti per le funzioni analitiche uniformi sopra una curva algebrica. Ann. Scuola Norm. Super. Pisa (3) 5, 39–56 (1951).

The linear analytic functionals of Fantappiè are defined on single-valued analytic functions $y(t)$. This paper considers such functionals on spaces of functions analytic on

algebraic curves. An algebraic curve C is any of the class of algebraic curves equivalent under birational transformations. Neighborhoods of a branch point are defined by the transformation $\tau=(t-t_0)^{1/r}$. The function $y(t)$ on the curve C is analytic at t_0 if it is analytic as a function of τ at $\tau=0$. There is introduced the notion of an analytic differential form on C, viz. $u(\alpha)d\alpha$, where $u(\alpha)$ is analytic on C and $u(\alpha)d\alpha$ is invariant for points on overlapping neighborhoods of C. The notions of an analytic functional space S_C and linear base (A) of functions regular on a closed set A of C carry over, as well as that of an analytic line $y(t, \alpha)$, except that y is assumed here to be analytic in t and α on the product space of algebraic curves C, G: $C\times G$. An indicatrix of the identity is a differential analytic line $v(t, \alpha)d\alpha$, t and α on C, relative to region H of C, if, for every t_0 of H, $v(t_0, \alpha)d\alpha$ is a regular differential form for all points of H excepting t_0 and has a residue equal to l along any line λ in H containing a region of H and t_0. Then if F is a linear analytic functional on a linear base (A), H is a region containing the closed set A of the base (A), and $v(t, \alpha)d\alpha$ is an indicatrix of the identity, then $F[y(t)]=(2\pi i)^{-1}\int y(\alpha)F_t[v(t, \alpha)d\alpha]$, i.e., $u(\alpha)d\alpha=F_t[v(t, \alpha)d\alpha]$ is an indicatrix for F. Indicatrices of the identity are determined by the integrands of Abelian integrals of the 3rd kind on the algebraic curve C having two logarithmic singularities M and Q with polar periods $2\pi i$ and $-2\pi i$, Q being fixed and M with abscissa t, varies, the corresponding differential form $r(t, \alpha)d\alpha$ vanishing at a certain group of p points of C different from M and Q.

T. H. Hildebrandt (Ann Arbor, Mich.).

Carafa, Mario. Espressione in forma finita di ogni funzionale analitico non lineare. Rend. Accad. Naz. dei XL (4) 1, 93–111 (1950).

This note is concerned with obtaining an expression for an analytic functional F of analytic functions $y(t)$ in a functional region R as defined by Fantappiè [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (6) 3, 453–683 (1930)]. Starting from the Fantappiè series for such a functional:

$$F[y_0(t)+u\varphi(t)]=F[y_0(t)]+\sum_{n=1}^{\infty}\frac{1}{(2\pi i)^n}\int\frac{d\alpha_1}{\alpha_1}\int\frac{d\alpha_2}{\alpha_2}\cdots$$
$$\times\int\frac{d\alpha_n}{\alpha_n}F^{(n)}\left[y_0(t);\frac{1}{\alpha_1},\cdots,\frac{1}{\alpha_n}\right]\varphi(\alpha_1)\cdots\varphi(\alpha_n),$$

where

$$F^{(n)}[y_0(t);\alpha_1,\cdots,\alpha_n]=$$
$$\frac{\partial^n}{\partial\epsilon_1\cdots\partial\epsilon_n}F\left[y_0(t)+\frac{\epsilon_1}{1-t\alpha_1}+\cdots+\frac{\epsilon_n}{1-t\alpha_n}\right]_{\epsilon_1=\cdots=\epsilon_n=0},$$

the author introduces a set of special functions and transformations and shows that it is possible to represent F in some neighborhood of $y_0(t)$ in the form

$$F(y(t))=Z\left[\frac{v}{1-\epsilon Q(y(t)-y_0(t))}\right],$$

where Z and Q are linear functionals independent of F, and the function v, called a local indicatrix relative to y_0, depends on y_0 and the Fantappiè expansion of F for a special function φ.

T. H. Hildebrandt (Ann Arbor, Mich.).

Riesz, Frédéric. Sur la représentation des opérations fonctionnelles linéaires par des intégrales de Stieltjes. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] 21, no. 16, 5 pp. (1952).

This note adds another proof of the author's theorem for the representation of the most general linear continuous functional on the space of continuous functions on an interval (a, b) as a Stieltjes integral $\int_a^b f\,d\alpha$. He proves first that if the space of continuous functions is normed by $\|f\|=\int_a^b|f|$ and B is a linear continuous functional on this space, extended to include the step functions, then $\beta(x)$ defined as the value of B for the characteristic function of the interval (a, x) satisfies the Lipschitz condition

$$|\beta(x_2)-\beta(x_1)|<\|B\|\cdot|x_2-x_1|$$

and $B(f)=\int_a^b f\,d\beta$. Then $A(f)$ is obtained as the limit of a sequence of functionals of the type B by the device:

$$A(f)=\lim_n A_n(f)=\lim_n nB\left[F\left(x+\frac{1}{n}\right)-F(x)\right]$$

where $F(x)=\int_a^x f(t)dt$, f being extended by symmetry beyond b.

T. H. Hildebrandt (Ann Arbor, Mich.).

√ ***Schwartz, Laurent. Théorie des noyaux.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1 pp. 220–230. Amer. Math. Soc., Providence, R. I., 1952.

This is an exposition without proofs of the author's theory of mappings between spaces of distributions on euclidean space [see the author's Théorie des distributions, tome I, Actualités Sci. Ind., no. 1091, Hermann, Paris, 1950; these Rev. 12, 31], or equivalently, of bilinear functionals on such spaces. A continuous linear mapping $f\to g$ between Banach spaces can in many special cases, but by no means always, be represented in terms of a kernel K, so that (in the case of spaces of functions, and with an appropriate notion of integral) $g(x)=\int K(x, y)f(y)dy$. The basis of the present work is the fact that such a kernel will always exist for mappings between spaces of distributions if it is not required to be a function, but is itself allowed to be a distribution; this type of representation for mappings had previously been used in many special cases in which a function-kernel does not exist, e.g. for mappings of Hilbert space into itself. The author treats among other aspects the ways in which: a) compactness properties of the kernel are reflected in properties of the corresponding mapping; b) differentiability properties of images under the mapping are connected with smoothness properties of the kernel. The kernel of the product of two mappings is expressible as the composition $\int K_1(x, y)K_2(y, z)dy$ (suitably interpreted) of the corresponding kernels K_1 and K_2. The results are not restricted to the basic spaces $\mathfrak{D}$, $\mathfrak{D}'$, $\mathcal{E}$, $\mathcal{E}'$ employed by the author but are applicable to most of the commonly-used function spaces also, by virtue of a technique for dealing with mappings between elements of an appropriate family of locally convex spaces $\mathfrak{A}$ such that $\mathfrak{D}\subset\mathfrak{A}\subset\mathfrak{D}'$.

I. E. Segal (Chicago, Ill.).

Ficken, F. A. The continuation method for functional equations. Comm. Pure Appl. Math. 4, 435–456 (1951).

As is well known, the so-called continuation method for solving equations consists essentially in the following: let $T(x)=0$ be the equation to be solved in some x-domain D. Suppose there exists a family of equations $T_s(x)=0$ $(0\leqq s\leqq 1)$ such that (i) $T_1(x)=T(x)$, (ii) $T_0(x)=0$ has a solution $x=x_0$ in D, and (iii) there exists a $\delta>0$ such that if s' is a fixed value in the interval $0\leqq s'<1$ for which $T_{s'}(x)=0$ has a solution $x=x'$, then the equation $T_s(x)$ has one and only one continuous solution $x=x(s)$ with $x(s')=x'$ for all s in $s'\leqq s\leqq\min(1, s'+\delta)$. Under these conditions one can obviously "continue" the solution x_0 in a finite number of steps to a solution $x_1=x$ of the equation $T_1(x)=T(x)=0$.

The main object of the present paper is to establish explicitly and in detail conditions sufficient for this process to be carried through in the case that the "x-domain" D is the sphere $\|x\| \leqq N$ in a Banach space S, and $T(x)$ represents a map of this sphere into S. The conditions given are also sufficient for uniqueness. These conditions are of two kinds: conditions "in the small" concerning uniform continuity, differentiability and non-singularity assumptions on the one hand, and a global "a priori" condition on the other, stating that those $x' \epsilon D$ which for some s in $0 \leqq s \leqq 1$ satisfy $T(x', s) = 0$ are uniformly bounded away from the boundary $\|x\| = N$ of D, i.e. that there exists a positive ρ such that $\|x'\| \leqq N - \rho$.

Concerning the historical remarks in section I (p. 437, B) of the paper it should be pointed out that it was Leray (on pp. 250 and 258 of the paper listed as no. 22 in the bibliography [Comment. Math. Helvetici **8**, 149–180, 250–263 (1935)]) who first applied his and Schauder's topological method also to uniqueness theorems. *E. H. Rothe.*

✓★**Halmos, Paul R. Introduction to Hilbert Space and the Theory of Spectral Multiplicity.** Chelsea Publishing Company, New York, N. Y., 1951. 114 pp. \$3.25.

The main purpose of this book is to present the so-called multiplicity theory and the theory of unitary equivalence, for arbitrary spectral measures, in separable or not separable Hilbert space. This is developed in chapter III; the preceding parts serve as an introduction. Chapter I deals with the "geometry" of Hilbert space: subspaces, orthogonal decompositions, bounded linear, bilinear and quadratic forms. In chapter II, one defines, after a careful study of projections, the concept of spectral measure: X being a set with a specified Boolean σ-algebra S of subsets, a spectral measure in X is a projection-valued function $E(M)$, defined for each $M \epsilon S$, such that $E(X) = I$ and $E(\bigcup_n M_n) = \sum_n E(M_n)$ whenever $\{M_n\}$ is a disjoint sequence of sets in S. Such a function is always monotone, subtractive, modular and multiplicative, i.e. $E(M) \leqq E(N)$, $E(N-M) = E(N) - E(M)$ if $M \subseteq N$,

$$\text{and } E(M \cap N) + E(M \cup N) = E(M) + E(N),$$
$$E(M \cap N) = E(M)E(N)$$

if M, N are arbitrary. A spectral measure defined on the class of all Borel sets in the complex plane, or in the real line, is called a complex or real spectral measure, respectively. The spectral theorem for bounded Hermitian operators is proved by reduction to the moment problem

$$(A^n x, y) = \int_m^M \lambda^n d\mu_\lambda(x, y) \quad (n = 0, 1, 2, \cdots)$$

[cf. F. Riesz, Nachr. Ges. Wiss. Göttingen **1910**, 190–195; W. F. Eberlein, Bull. Amer. Math. Soc. **52**, 328–331 (1946); these Rev. **7**, 453]. The spectral theorem for normal operators is derived from that one for Hermitian operators. Fuglede's recently found theorem, asserting that, for a normal operator $A = \int \lambda dE_\lambda$ (where E_λ is the complex spectral measure corresponding to A) and for an operator B, $AB = BA$ implies $E_\lambda B = BE_\lambda$, is proved by the author's own method [Acta Sci. Math. Szeged **12B**, 153–156 (1950); these Rev. **11**, 600]. Unbounded operators are not considered.

Chapter III deals with the multiplicity theory of spectral measures, initiated by Hellinger and Hahn for bounded Hermitian operators in separable Hilbert space, and developed by F. Wecken [Math. Ann. **116**, 422–455 (1939)], H. Nakano [Ann. of Math. (2) **42**, 657–664 (1941); Math. Ann. **118**, 112–133 (1941); these Rev. **3**, 51; **4**, 13] and Plessner and Rohlin [Uspehi Matem. Nauk (N.S.) **1**, no. 1(11), 71–191, 192–206 (1946); these Rev. **9**, 43], for bounded or not bounded self-adjoint and normal operators in separable or not separable Hilbert space. The approach to this theory, as presented by the author, has much claim to novelty. By a skillful permutation of the fundamental ideas of Wecken and Nakano, and constantly referring to the simple situation in the finite-dimensional case, the author succeeds in presenting the theory in a clear and perspicuous form. Since he does not restrict himself to real or complex spectral measures (as Wecken and Nakano did), the theory as it appears includes not only the multiplicity theory of (bounded or not bounded) normal operators, but also the multiplicity theory of unitary representations of locally compact abelian groups, and the multiplicity theory of an arbitrary weakly closed, self-adjoint, commutative operator algebra. The book concludes with a two page bibliography mentioning only the recent contributions (and not mentioning the names of Hellinger, Hahn and Friedrichs). *B. Sz.-Nagy* (Szeged).

Halmos, Paul R. Commutators of operators. Amer. J. Math. **74**, 237–240 (1952).

If P and Q are operators on complex Hilbert space H (i.e. bounded linear transformations of H into itself), the commutator of P and Q is defined by $[P, Q] = PQ - QP$ and the self commutator $[P]$ by $[P^*, P]$. Theorem I: Every hermitian operator on an infinite-dimensional Hilbert space is the sum of two self commutators. Theorem II: Every hermitian operator on an infinite-dimensional Hilbert space is the real part of a commutator. Corollary: There exist operators P and Q such that $\inf \{ |([P, Q]x, x)| ; \|x\| = 1\} = 1$. The corollary answers a question raised by A. Wintner [Physical Rev. (2) **71**, 738–739 (1947); these Rev. **8**, 589] and discussed by C. R. Putnam [same J. **73**, 127–131 (1951); these Rev. **12**, 836]. I. Kaplansky comments that theorem I can be used to prove that the concept of trace cannot be extended to all operators on infinite-dimensional Hilbert spaces. Previous results of this nature used assumptions of continuity or positiveness of trace.

S. Sherman (Sherman Oaks, Calif.).

Heinz, Erhard. Zur Theorie der Hermiteschen Operatoren des Hilbertschen Raumes. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. Math.-Phys.-Chem. Abt. **1951**, no. 2, 4 pp. (1951).

A point of condensation λ for a self-adjoint operator A with resolution of identity $E(\lambda)$ is real number such that $E(\lambda+\delta) - E(\lambda-\delta)$ has an infinite-dimensional range for every $\delta > 0$. Let H be a Hermitian operator with deficiency indices (m, m), $m < \infty$. If A is any self-adjoint extension of H, the range of $E(b) - E(a)$ differs by at most an m-dimensional manifold from its intersection with the domain of H. Thus every self-adjoint extension of H has the same condensation points. The author also establishes that if H is Hermitian and for a real λ_0, $H - \lambda_0$ has a bounded inverse then the deficiency indices of H are equal. This is equivalent to a result of Calkin, that under these circumstances H has a self-adjoint extension. *F. J. Murray.*

Karush, William. Determination of the extreme values of the spectrum of a bounded self-adjoint operator. Proc. Amer. Math. Soc. **2**, 980–989 (1951).

Let A be a bounded selfadjoint operator on a real Hilbert space. Denote by $[A(x)]$ the subspace spanned by $A^n x$ for all n, by $[A(x)]_s$ that spanned by $A^n x$ for $0 \leqq n \leqq s-1$, where s is an integer greater than 1. The upper bound λ of

the restriction B of A to $[A(x_0)]$ can be determined by the following iterative method. Given x_i, let μ_i be the supremum of $(Ax, x)/\|x\|^2$ for $x \varepsilon [A(x_i)]_s$, and $x_{i+1} \varepsilon [A(x_i)]$, an element of form $x_i + \eta$, with η orthogonal to x_i, for which this supremum is attained. Then $\mu_i \to \lambda$ as $i \to \infty$, and if λ is a characteristic value of A, $x_i/\|x_i\|$ tends weakly to a characteristic vector; if not, it tends to 0. *J. L. B. Cooper.*

Zaanen, A. C. Normalisable transformations in Hilbert space and systems of linear integral equations. Acta Math. **83**, 197–248 (1950).

In this paper the author extends his earlier work on symmetrisable operators in Hilbert space to what he calls 'normalisable' operators. He considers a Hilbert space $\mathfrak{R}$, which need not be separable or complete, and a bounded non-negative definite self-adjoint operator H in $\mathfrak{R}$, with the additional property (which it has in any case if $\mathfrak{R}$ is complete) that every element of $\mathfrak{R}$ can be expressed as the sum of an element in the null-space $[\mathfrak{L}]$ of H and one orthogonal to $[\mathfrak{L}]$. The projection on the orthogonal complement $[\mathfrak{M}]$ of $[\mathfrak{L}]$ is denoted by E. Two bounded linear operators K and $\tilde{K}$ are said to be H-adjoints if

$$(HKf, g) = (Hf, \tilde{K}g) \qquad (f, g \varepsilon \mathfrak{R});$$

if K has an H-adjoint $\tilde{K}$ and $HK\tilde{K} = H\tilde{K}K$, K is said to be normalisable (relative to H). Two elements f and g are said to be H-orthogonal if $(Hf, g) = 0$, and a sequence (φ_i) is H-orthonormal if $(H\varphi_i, \varphi_j) = \delta_{ij}$; the pseudo-norm $(H f, f)^{\frac{1}{2}}$ is denoted by $N(f)$.

The main result of the paper is as follows. Let K be a normalisable operator in $\mathfrak{R}$ such that $T = EK$ is completely continuous and $P = HK \neq 0$; then T has H-adjoint $\tilde{T} = E\tilde{K}$. There is an H-orthonormal sequence (φ_i) and a sequence of corresponding eigen-values (λ_i) such that

$$T\varphi_i = \lambda_i \varphi_i, \quad \tilde{T}\varphi_i = \bar{\lambda}_i \varphi_i \qquad (i = 1, 2, \cdots).$$

If $f \varepsilon \mathfrak{R}$ and $\alpha_i = (Hf, \varphi_i)$, then

$$N(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i) \to 0 \qquad (n \to \infty),$$

$$N(\tilde{K}f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i) \to 0 \qquad (n \to \infty).$$

If $Hf = 0$ implies $Kf = 0$, this expansion theorem can be improved. There is then an H-orthonormal sequence (φ_i) and a sequence of eigenvalues (λ_i) such that

$$T\varphi_i = \lambda_i \varphi_i \qquad (i = 1, 2, \cdots).$$

If $f \varepsilon \mathfrak{R}$ and $\alpha_i = (Hf, \varphi_i)$, then the expansions hold in the same sense as before. When $H = I$, these reduce to known results for normal operators [F. Rellich, Math. Ann. **110**, 342–356 (1934)].

If, in particular, $K = AH$, where A is a bounded linear operator, then K and $\tilde{K} = A^*H$ are H-adjoints; if

$$HAHA^*H = HA^*HAH,$$

then K is normalisable. If either A or H is completely continuous, the expansion theorem can be further improved; we then have

$$Kf = \sum_i \lambda_i \alpha_i \varphi_i + h,$$

convergence being in the sense of the ordinary norm, and h being in the null-space of H. When $n \geqq 2$,

$$K^n f = \sum_i \lambda_i^n \alpha_i \varphi_i,$$

with no remainder term. The author also discusses the relation of his normalisable operators with normal operators in the quotient space of $\mathfrak{R}$ by the equivalence relation $Hf = Hg$, and the effect of going over into the completed space $\bar{\mathfrak{R}}$. A version of the theory of singular functions and singular values is obtained for operators possessing an H-adjoint. The paper concludes with some applications to integral equations and systems of integral equations.

F. Smithies (Cambridge, England).

Reid, William T. Symmetrizable completely continuous linear transformations in Hilbert space. Duke Math. J. **18**, 41–56 (1951).

All transformations of Hilbert space $\mathfrak{H}$ dealt with are everywhere defined, linear and bounded. The following inequality is basic for the discussion: If K and S are such that S and SK are symmetric and $S \geqq 0$, then

$$|(x, SKx)| \leqq \|K\|(x, Sx)$$

for all $x \varepsilon \mathfrak{H}$. Using this inequality and the general solvability theorems for completely continuous (c.c.) transformations, it is proved, by means of an extremizing process that is a direct extension of one frequently used for symmetric transformations, that if K is c.c. and symmetrizable by a non-negative transformation S, and if $SK \neq 0$, then K has a real proper value. Symmetrizability means that SK is symmetric. If the symmetrizability is "full" in the sense that $Sx \neq 0$ for each proper element x of K corresponding to a non-zero proper value, then there exists a maximal set of linearly independent proper elements u_n corresponding to non-zero proper values λ_n, with $|\lambda_1| \geqq |\lambda_2| \geqq \cdots$, such that $(u_m, Su_n) = \delta_{mn}$, $SKx = \sum \delta_n(x, Su_n)Su_n$. It is noted that for a special class of symmetrizable c.c. transformations the spectral theory may be derived without the use of the general solvability theorems for c.c. transformations, this class containing as special instances the transformations K of the following two types: (i) $K = GS^{\frac{1}{2}}$ where G is c.c., $S \geqq 0$, K is symmetrizable by S; (ii) $K = GS$ where G is c.c., S is symmetric, K is symmetrizable by S, and $SK^p \geqq 0$ for some $p \geqq 0$. There follows an extension of a result of Rellich [Math. Ann. **110**, 342–356 (1934)] on c.c. normal transformations. The results of the paper are compared with the previous work of A. C. Zaanen [Nieuw Arch. Wiskunde (2) **22**, 57–80 (1943); Nederl. Akad. Wetensch., Proc. **49**, 194–204, 205–212 (1946); these Rev. **7**, 453, 621 **8**, 28; and the paper reviewed above]. *B. Sz.-Nagy* (Szeged).

Taldykin, A. T. On linear equations in Hilbert space. Mat. Sbornik N.S. **29(71)**, 529–550 (1951). (Russian)

Let A be an operator, defined in a Hilbert space H, which can be expressed in the form $A = V + B$, where V is totally continuous, B has a continuous inverse. Then $r_A = \sup \lambda$ for all λ with $\sum \lambda^n B^n$ convergent is called the Fredholm radius of A. The region in which the Fredholm theory concerning the solubility of $T_\lambda x = y$, $T_\lambda^* x = y$ $(T_\lambda = E - \lambda A)$ holds is called Ω_A, the Fredholm region; $|\lambda| < r_A$ is called the Fredholm circle, and is inside the Fredholm region. The paper is concerned with values of λ in Ω_A.

Let $\lambda S_\lambda = T_\lambda^{-1} - E$. At a pole λ_0 of S_λ, $S_\lambda = \gamma_\lambda + H_\lambda$, where $\gamma_\lambda = \sum_{k=-m}^{-1} (\lambda - \lambda_0)^k C_k$, $H_\lambda = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k C_k$. Then $\gamma_\lambda H_\mu = H_\mu \gamma_\lambda = 0$, and $C_{-m}h$ is a characteristic vector of A for λ_0. The characteristic vectors of A and A^* for different λ are orthogonal. If all poles of S_λ are simple, then to each characteristic vector of A corresponds one of A^*, not orthogonal to it; and if h is orthogonal to all characteristic vectors of A^*, $h + \lambda S_\lambda h$ has no singularities in Ω_A. If $A = V + B$

where $V+B$ is a resolution of the type defined above, and such that $\|A\|>\|B\|$, and if A^*A-AA^* is semidefinite, then A has at least one characteristic value in the Fredholm circle. If all characteristic values of A are simple, and h is orthogonal to all its characteristic vectors, then $Ah=A^*h=0$. This occurs in particular if A is normal.

Green's functions in one dimension which define operators satisfying these conditions and expansions in series of the solutions of the Fredholm equation are discussed.

J. L. B. Cooper (Cardiff).

Horn, Alfred. On the singular values of a product of completely continuous operators. Proc. Nat. Acad. Sci. U. S. A. **36**, 374–375 (1950).

The author presents an elementary proof of the following theorem, which was first proved by function-theory methods by S. H. Chang [Trans. Amer. Math. Soc. **67**, 351–367 (1949); these Rev. **11**, 523]. If $K=K_1K_2\cdots K_m$, where each K_i is an operator of finite norm in Hilbert space, and (γ_i) is the sequence of inverse singular values of K, then $\sum_i\gamma_i^{2/m}$ is convergent. In the course of the proof, he shows that if H is a non-negative definite completely continuous operator whose first n eigen-values are $\lambda_1, \lambda_2, \cdots, \lambda_n$, and $y_1, y_2, \cdots, y_n$ are arbitrary elements of Hilbert space, then

$$\det[(Hy_i, y_j)]\leqq\lambda_1\lambda_2\cdots\lambda_n \det[(y_i, y_j)].$$

F. Smithies (Cambridge, England).

Pini, Bruno. Su una classe di forme quadratiche dello spazio hilbertiano. Giorn. Mat. Battaglini (4) **4**(80), 129–141 (1951).

A represents a limited symmetric matrix on the sequential Hilbert space: $\sum_n|x_n|^2<\infty$. It is shown that a necessary and sufficient condition that A be positive definite ($(x, Ax)\geqq 0$ for all x, zero only for $x=0$) is that the principal minors; $\det a_{ij}$, $i, j=1, \cdots, n$, be positive for all n and that the rows of A form a complete set of vectors for the Hilbert space. It is also shown that the properties of limitedness, complete continuity, sum of squares of elements convergent, and positive definiteness carry over from A to the matrix $A^{(k)}$ for all k. The matrix $A^{(k)}$ is formed of the kth order minors of A. If

$$\begin{pmatrix} i_1\cdots i_k \\ j_1\cdots j_k \end{pmatrix}$$

is the determinant of a_{ij} with $i=i_1<\cdots<i_k$, $j=j_1<\cdots<j_k$ then $i_1\cdots i_k$ is fixed for any row of $A^{(k)}$ and $j_1\cdots j_k$ for any column, the diagonal elements being those for which $i_1=j_1, \cdots, i_k=j_k$.

T. H. Hildebrandt.

Turumaru, Takasi. On the commutativity of the C^*-algebra. Kōdai Math. Sem. Rep. **1951**, 51 (1951).

A self-adjoint algebra A of bounded operators on a (real or complex) Hilbert space is commutative provided the following related algebra is associative: the self-adjoint elements of A with multiplication defined by the equation $x\circ y=\frac{1}{2}(xy+yx)$. In the case of a complex space this follows from a result of Kaplansky [Bull. Amer. Math. Soc. **54**, 575–580 (1948), esp. p. 580; these Rev. **10**, 7] and also from a result of Segal [Ann. of Math. (2) **48**, 930–948 (1947), Theorem 1; these Rev. **9**, 241].

I. E. Segal.

Ghika, Al. The extension of general linear functionals in semi-normed modules. Acad. Repub. Pop. Romåne. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 399–405 (1950). (Romanian. Russian and French summaries)

The author defines an F-ordered ring A to be a lattice-ordered ring with unity element ϵ such that: $\epsilon>0$; if $\alpha\epsilon A$ is not a divisor of 0, then there exists its inverse $\alpha^{-1}\epsilon A$; if δ is a divisor of 0, there exists α and α^{-1} in A such that $\delta(\alpha-\delta)=0$; if α, β, γ are in A then

$$\alpha+\sup(\beta, \gamma)=\sup(\alpha+\beta, \alpha+\gamma);$$

if, moreover, $\alpha\geqq 0$, then $\alpha\sup(\beta, \gamma)=\sup(\alpha\beta, \alpha\gamma)$. A number of consequences of this definition are then enumerated, e.g., A order-contains the reals, and every idempotent of A is the inf of a sequence of elements in A which have inverses. The author further considers unitary A-modules $(E, A)=E$ [see N. Bourbaki, Eléments de mathématique, VI: Livre II, Chapitre II, Hermann, Paris, 1947, p. 3; these Rev. **9**, 406] with a function ("semi-norm") p on E to A such that x and y in E, $\alpha\epsilon A$ imply $p(x+y)\leqq p(x)+p(y)$, $p(\alpha x)=|\alpha|p(x)$ (where $|\alpha|=\sup(\alpha, 0)+\sup(-\alpha, 0)$). It is then asserted that the proof of the Hahn-Banach theorem as, for instance, in Banach's book [Théorie des opérations linéaires, Warsaw, 1932, p. 28] implies the truth of the analogous assertion: if E is an A-module as described above and if there exists a sub-A-module $G=(G, A)$ of E and an A-linear mapping f of G into A such that $f(x)\leqq p(x)$, then f can be extended to a similarly bounded F mapping E A-linearly into A, and there exist nontrivial such F.

G. K. Kalisch (Minneapolis, Minn.).

Ghika, Al. On ordered commutative rings. Acad. Repub. Pop. Romåne. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 509–517 (1950). (Romanian. Russian and French summaries)

This note gives an alternative set of axioms for F-ordered rings A [see preceding review] one of whose conditions states that if δ is a divisor of 0 there exists an idempotent element η and an element α with an inverse such that $\delta=\eta\alpha$. The principal result of the paper is the establishment of a necessary and sufficient condition that A be algebraically and order isomorphic with a ring of real-valued functions (ordered point-wise) on some set T: given an idempotent element $\omega\neq 0$ of A, there exists a minimal idempotent element ω' such that $0<\omega'\leqq\omega$. Then T turns out to be the set of minimal idempotents. [Note: the condition mentioned in the first sentence of this review insures semisimplicity; this together with the fact that, if ω is in T, then ωA is algebraically and order isomorphic to the reals forms the basis of the proof; see also Vernikoff, Krein, and Tovbin, C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 785–787 (1941); these Rev. **2**, 314.]

G. K. Kalisch.

Theory of Probability

O ★**Gnedenko, B. V. Kurs teorii veroyatnosteĭ.** [**Course in the Theory of Probability**]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 387 pp.

This book is intended as a first introduction to probability and statistics. It is written in a clear and concise manner. Radical trimming and streamlining enable the author to discuss in a short space a variety of modern topics. This will greatly increase the usefulness of the book, although it is unavoidable that the beginner will occasionally not see the proper relations between the several parts. The rapid succession of different landscapes leaves no time to the beginner to get familiarized with any one in particular, but this initial bewilderment is counterbalanced by the fact that the book will remain a useful companion even after the first maturing process.

In the case of a real Hilbert space the conclusion is false as stated, and what the author really proves is as follows: the self-adjacent elements of A commute.

In the exposition the whole emphasis is laid on the analytical part, and the probabilistic interpretation and background are cut short. Moreover, the book covers only topics which can be expressed in terms of distribution functions. Probability is introduced as a completely additive set function, but this axiomatic background is nowhere used. Strong limit laws are not treated, and the theory of stochastic processes is restricted to transition probabilities and correlations without mentioning the sample functions or the difficulties concerning conditional probabilities.

Chapter 1 (48 p.) follows the conventional line of discussing special examples, including Buffon's needle problem. The modern touch is given by describing the axiomatic setup and the introduction of the simplest problem for Geiger-Müller counters. Chapter 2 (38 p.) covers the polynomial distribution and gives a detailed derivation of the normal approximation to it. Moreover, it introduces the Poisson distribution, and briefly mentions random walk problems. The six pages of chapter 3 are devoted to finite Markov chains, excluding periodic and transient states. Chapter 4 (32 pp.) introduces distribution functions and convolutions, chapter 5 (22 pp.) moments and mixed moments. There follows in chapter 6 (20 pp.) the weak law of large numbers and Kolmogorov's form of the strong law (which, however, is formulated in terms of finite collections of distributions rather than of measure). Chapter 7 (28 pp.) is devoted to characteristic functions in one and more dimensions, Bochner's theorem, etc. Chapter 8 (15 pp.) contains the Lindeberg and Lyapunov forms of the central limit theorem and a special discussion for discrete distributions. No error estimates are given. Chapter 9 (19 pp.) is devoted to infinitely divisible distributions (without mention of stable distributions). Chapter 10 (33 pp.) discusses Markov processes. The two diffusion equations and the corresponding equations for completely discontinuous processes are derived following Kolmogorov and the reviewer. There follows a discussion of processes with independent increments in connection with infinitely divisible distributions and finally Hinčin's representation of the autocorrelation of a stationary process. The last chapter (55 pp.) gives an introduction to statistics. It contains theorems of Glivenko, Kolmogorov, and Smirnov on empirical distributions, estimation of parameters, the notions of testing statistical hypotheses, and of sequential analysis. A 28 page historical survey and tables conclude the book. The many references to Marx, Engels, Lenin, and Stalin are a deplorable sign of our times.

W. Feller (Princeton, N. J.).

Kunisawa, Kiyonori. A remark on the dispersion. Kōdai Math. Sem. Rep. 1951, 71–72 (1951).

If $F(x)$ is a distribution function let $\bar{F}(x)$ be the symmetrized distribution,

$$\Psi_F(l)=\int_{-\infty}^{+\infty}\frac{l^2}{l^2+x^2}d\bar{F}(x),$$

and $D_F(\alpha)$ the inverse function of $\alpha=\Psi_F(l)$. Let x_k be a sequence of mutually independent random variables with distributions F_k and variances σ_k^2. Put $s_n^2=\sigma_1^2+\cdots+\sigma_n^2$ and let $\frac{3}{4}<\alpha<1$. Finally, let $D_n(\alpha)=D_{F_1*\cdots*F_n}(\alpha)$. The author proves that the ratios $s_n^2/D_n(\alpha)$ and $D_n(\alpha)/s_n^2$ are bounded if and only if

$$\sum_1^n\int_1^{\infty}x\{1-F_k(D_n(\alpha)x\}dx$$

is uniformly bounded. *W. Feller* (Princeton, N. J.).

Bellman, Richard, and Harris, Theodore. Recurrence times for the Ehrenfest model. Pacific J. Math. 1, 179–193 (1951).

Suppose that $2N$ balls are initially divided between two urns and that during any small time-interval of length h each ball has probability $\frac{1}{2}h+o(h)$ of changing urns and probability $1-\frac{1}{2}h+o(h)$ of remaining in its urn. The balls are statistically independent. This is the time-continuous analog of the discrete Ehrenfest model, which was discussed in detail by Kac [Amer. Math. Monthly **54**, 369–391 (1947); these Rev. **9**, 46]. Denote the number of balls in the first urn as the state of the system and let L_{jk} be the first passage time from state j to state k (so that L_{kk} is the recurrence time of k). Let $m_k=E(L_{Nk})$. Using Laplace transforms the authors show that when $N\rightarrow\infty$ and the ratio k/N remains bounded away from 1 the distribution function of the random variable L_{kk}/m_k converges to $1-e^{-x}$ (for $x>0$). On the other hand, if $m_k=E(L_{kk})$ and $k/N\rightarrow\gamma<1$, then for every $a>0$ we have

$$\lim_{N\rightarrow\infty}\sup_{u\geq a}|P_r\{L_{kk}>um_k\}-\gamma\exp(-\gamma u)|=0.$$

Finally the authors also consider the passage to the limit $(k-N)N^{-1/2}\rightarrow x$, where the process approaches a Gaussian process. Here the limiting form of the Laplace transform of $L_{N,k}$ is given. *W. Feller* (Princeton, N. J.).

Tsuchikura, Tamotsu. On the function $t-[t]-\frac{1}{2}$. Tôhoku Math. J. (2) 3, 208–211 (1951).

Let $f(t)=t-[t]-\frac{1}{2}$ and a be an integer $\geqq 2$. Let $\epsilon_k(t)$ be the kth digit in the a-dic expansion of t, $\delta_k(t)=\epsilon_k(t)-(a-1)/2$. Then a simple computation shows that

$$\sum_{i=0}^{N}f(a^i)=(a-1)^{-1}\sum_{k=1}^{N}\delta_k(t)+O(1).$$

Applying the law of the iterated logarithm we obtain $\limsup_{N\rightarrow\infty}(\sum_{i=0}^{N}f(a^it))/N(\log\log N)^{1/2}=[(a+1)/b(a-1)]^{1/2}$ for almost all t. Using a simple corollary of a result of the reviewer and Erdős [Ann. of Math. (2) **48**, 1003–1013 (1947); these Rev. **9**, 292] we obtain

$$\liminf_{N\rightarrow\infty}|\sum_{i=0}^{N}f(a^it)|\leqq(a-1)^{-1}(|f(t)|+\tfrac{1}{2})$$

for almost all t. These results improve upon crude results of Koksma [Nieuw Arch. Wiskunde (2) **21**, 242–267 (1943); these Rev. **7**, 369] obtained by non-probabilistic methods. A category theorem is also proved which supplements the strong law of large numbers applied to the case on hand.

K. L. Chung (Ithaca, N. Y.).

Takano, Kinsaku. On the convergence of classes of distributions. Ann. Inst. Statist. Math., Tokyo 3, 7–15 (1951).

The principal result is another proof of the following theorem of Khintchine [Izvestiya Naučno-Issled. Inst. Mat. Meh. Tomsk. Gosud. Univ. 1, 258–262 (1937)]: Let $F_n(x)$, $n=1, 2, \cdots$ be a sequence of distribution functions. Suppose that there exist sequences of positive numbers $\{a_n\}$, $\{\alpha_n\}$, and sequences of real numbers $\{b_n\}$ and $\{\beta_n\}$ with the property that $\lim_{n\rightarrow\infty}F_n(a_nx+b_n)=\Phi(x)$, $\lim_{n\rightarrow\infty}F_n(\alpha_nx+\beta_n)=\Psi(x)$, at every continuity point of $\Phi(x)$ or $\Psi(x)$, respectively, where $\Phi(x)$ and $\Psi(x)$ are distribution functions. Then, if neither $\Phi(x)$ nor $\Psi(x)$ assigns probability one to a single point, there exist the limits

$$\lim_{n\rightarrow\infty}\frac{\alpha_n}{a_n}=A,\quad \lim_{n\rightarrow\infty}\frac{\beta_n-b_n}{a_n}=B$$

and, for all x,

$$\Psi(x)=\Phi(Ax+B).$$

The proof makes use of the inverse of a distribution function.
J. Wolfowitz (Ithaca, N. Y.).

Rosenblatt, M. On the oscillation of sums of random variables. Trans. Amer. Math. Soc. **72**, 165–178 (1952).

Soit $\{X_n\}$ une suite de variables aléatoires indépendantes de même fonction de répartition $F(x)$; $S_n=\sum_{j=1}^n X_j$; la suite $\{S_n\}$ oscille si $\Pr(S_n>0$ une infinité de fois$)=\Pr(S_n\leqq 0$ une infinité de fois$)=1$. a) Si $E(|X_n|)<+\infty$ et si $\Pr(X_n\neq 0)>0$, il faut et il suffit pour que $\{S_n\}$ oscille que $\{S_n\}$ admette des valeurs récurrentes finies autres que 0, c'est-à-dire que $E(X_n)=0$ d'après Chung et Fuchs [Mem. Amer. Math. Soc., no. 6 (1951); ces Rev. **12**, 722]. b) Dans le cas général une condition nécessaire et suffisante pour que $\{S_n\}$ oscille est que

$$\sum_i \Pr(S_i>0,\, S_{i+1}\leqq 0)=\sum_i \Pr(S_i\leqq 0,\, S_{i+1}>0)=+\infty;$$

l'auteur en déduit pour le cas où $E(|X_n|)=+\infty$ des conditions nécessaires, ou suffisantes, ou nécessaires et suffisantes, portant sur $F(x)$ et valables si $F(x)$ satisfait à certaines conditions.
R. Fortet (Caen).

Chung, K. L., and Erdös, P. On the application of the Borel-Cantelli lemma. Trans. Amer. Math. Soc. **72**, 179–186 (1952).

Borel's law of 0 or 1 for independent events is extended to a class of dependent events. It is then applied to sums S_n of independent, two-valued, and symmetric random variables: (1) If n_i is an increasing sequence of even integers such that $n_{i+1}-n_i>An_i^{\frac{1}{2}}$ then $P(S_n=0$ infinitely often$)=0$ or 1 according as $\sum n_i^{-\frac{1}{2}}$ is finite or infinite. (2) If φ_n increases with n and N_n is the number of positive S_k for $k\leqq n$, then $P(N_n\leqq n/\varphi_n$ infinitely often$)=0$ or 1 according as $\sum 1/n\varphi_n^{\frac{1}{2}}$ is finite or infinite.
M. Loève (Berkeley, Calif.).

√ ***Chung, Kai Lai. The strong law of large numbers.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 341–352. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Let $\{X_n\}$, $n=1, 2, \cdots$, denote a sequence of real-valued independent random variables. Let

$$\varphi_n=\sum_{k=1}^n X_k,\quad \varphi_n^0=\varphi_n-m(\varphi_n)$$

where $m(\varphi_n)$ is a median of φ_n. We say that the sequence $\{X_n\}$ satisfies the strong law of large numbers if

$$(1)\qquad P[\lim_{n\to\infty}\varphi_n^0/n=0]=1,$$

where $P[A]$ denotes the probability of the event A. At the present time no satisfactory necessary and sufficient conditions are known for the validity of (1); finding such conditions seems a difficult problem. The author gives an account of the recent attempts at solving this problem.

If the $\{X_n\}$ all have the same distribution $F(x)$, a well known theorem of Kolmogoroff states that a necessary and sufficient condition for (1) is that $\int_{-\infty}^{\infty}|x|dF(x)<\infty$. The author proves a more general theorem. Let $F(x,\theta)$ be a family of distribution functions depending on the parameter θ. Let $\{X_n\}$ be a sequence of random variables all having the same distribution function $F(x,\theta)$. The sequence $\{X_n\}$ obeys the strong law of large numbers uniformly with respect to θ if to every ϵ there exists an N_0 independent of θ so that

$$(2)\qquad P[|\varphi_n^0|\leqq n\epsilon \text{ for all } n\geqq N_0]\geqq 1-\epsilon$$

holds no matter what the value of θ is. A sufficient condition for (2) is that to every δ there is an $A(\delta)$ not depending on θ such that

$$(3)\qquad \int_{|x|>A(\delta)}|x|dF(x,\theta)<\delta.$$

(3) is also necessary if the median $m(\theta)$ is a bounded function of θ.

Several necessary and sufficient conditions are given for the strong law of large numbers but all are "unsatisfactory", as the author calls them, since the sum of the random variables enters into the condition. One of these is the following theorem due to Prohorov [Doklady Akad. Nauk SSSR (N.S.) **69**, 607–610 (1949); Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 523–536 (1950); these Rev. **11**, 375; **12**, 425]: A necessary and sufficient condition for (1) is that for every $\epsilon>0$

$$(4)\qquad \sum_{n=1}^{\infty}P[|(\varphi_{2^{n+1}}-\varphi_{2^n})^0|>2^n\epsilon]<\infty.$$

It is of course clear why the author calls (4) "unsatisfactory".

Finally, various satisfactory necessary and sufficient conditions are given for (1) if various conditions are imposed on the order of magnitude of the X_n.
P. Erdös.

Kawata, T., and Udagawa, M. On the strong law of large numbers. Kōdai Math. Sem. Rep. **1951**, 78–80 (1951).

Slight generalizations of Brunk's sufficient condition for the strong law of large numbers. The main proof is identical with one given (independently) by the reviewer in the paper reviewed above. An earlier announcement, with a somewhat different proof, was given by Prohorov [Izvestiya Akad. Nauk SSSR Ser. Mat. **14**, 523–536 (1950); these Rev. **12**, 425].
K. L. Chung (Ithaca, N. Y.).

√ ***Lévy, Paul. Processus à la fois stationnaires et markoviens pour les systèmes ayant une infinité dénombrable d'états possibles.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 549–554. Amer. Math. Soc., Providence, R. I., 1952.

The author considers Markov processes with a continuous time parameter and denumerably many possible states. Dropping all traditional measurability conditions he studies directly the sample functions. As a first step he examines the possible successions of states or paths; next he considers the time it takes to go through a given path. In this way he arrives at a classification of processes depending on whether discontinuities of the sample functions are (with probability one) well-ordered, etc. The interesting pathological possibilities are illustrated by examples and the general theory is indicated, but not fully described. The main results were given in C. R. Acad. Sci. Paris **231**, 467–468, 1208–1210 (1950); **232**, 1400–1402, 1803–1805 (1951); these Rev. **12**, 269, 619, 723, 840.
W. Feller (Princeton, N. J.).

√ ***Harris, T. E. Some mathematical models for branching processes.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 305–328. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

A detailed survey of existing theories and some open problems connected with branching processes. First the

now classical iterative scheme is described and the known limit theorems for the one-dimensional case (one type of particle) are enumerated. The author then treats the multi-dimensional case by matrix methods. Next the time-continuous analog is considered in the one-dimensional case. If such a process is considered only at times $t=0, 1, 2, \cdots$, then one gets the discrete branching process, but in general a given discrete process will not admit of a continuous interpolation. The author states a necessary and sufficient condition for the possibility of such an interpolation; this result seems not to have been published elsewhere. Finally, the author discusses the larger class of processes where the age of the particles plays a role. Several limit theorems are given in a new form. *W. Feller* (Princeton, N. J.).

***Snell, Laurie James. Applications of martingale system theorems.** Abstract of a Thesis, University of Illinois, 1951. i+5 pp.

A sequence x_n of chance variables is called a (generalized) semi-martingale if $E(|x_n|)<\infty$ for all n ($E(x_n)$ exists for all n) and $E(x_n \mid x_m) \geqq x_m$, $m<n$. For generalized semi-martingales, if b is the number of upcrossings of r_1, r_2 by $x_1, \cdots, x_m$, i.e. the number of pairs of integers (i_1, i_2) with $1 \leqq i_1 < i_2 \leqq m$ and $x_{i_1} < r_1$, $r_1 < x_i < r_2$ for $i_1 < i < i_2$ and $x_{i_2} > r_2$, then, with probability 1,

$$E(b \mid x_1) \leqq \frac{E(|x_m| \mid x_1) + r_1}{r_2 - r_1}.$$

A consequence of this result is the following theorem of Doob [Proceedings of the Second Berkeley Symposium on Statistics and Probability, 1950, Univ. of California Press, Berkeley and Los Angeles, 1951, pp. 269–277; these Rev. 13, 475]: if $x_1, x_2, \cdots$ is a semi-martingale with sup $E(|x_n|)<\infty$, then $x_n \to x^*$ with probability 1, and $E(|x^*|)<\infty$. An extension of this result to generalized semi-martingales is given and, as an application, a generalization of a theorem of Arrow, Girshick, and the reviewer [Econometrica 17, 213–244 (1949); these Rev. 11, 261] is obtained. No proofs are given. *D. Blackwell* (Washington, D. C.).

√*Kac, M. On some connections between probability theory and differential and integral equations. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 189–215. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

Various links between probability theory and differential and integral equations are well-known. The aim of the present expository lecture was not to give a systematic survey but to illustrate, on a variety of examples, some new cross-connections and to illustrate new methods which are due largely to the author. First the connection of probability limit theorems with the distribution of certain functionals connected with Markovian processes is described [cf. Kac, Trans. Amer. Math. Soc. 65, 1–13 (1949); these Rev. 10, 383 for the case of the Wiener-Bachelier process]. One is led to certain integral equations, which in particular cases reduce to differential equations. As applications the author considers the arc-sine law, problems connected with the Kolmogorov-Smirnov distribution [Proc. Nat. Acad. Sci. U. S. A. 35, 252–257 (1949); these Rev. 10, 614], and various representation problems. Of particular interest is the application to the ruin problem of processes which are related to symmetric stable distributions in the same way as the ordinary diffusion is related to the normal distribution. The general formulation of the problem is new [for the Cauchy process cf. Kac and Pollard, Canadian J. Math. 2, 375–384 (1950); these Rev. 12, 114].

So far differential and integral equations were applied to probability. Conversely, the author shows that probability can be used successfully (both heuristically and rigorously) to arrive at results about differential equations. In particular connections with the distribution of eigenvalues, the Weyl-Carleman theory, and potential theory are described. The arguments and methods cannot be reproduced in a few lines. *W. Feller* (Princeton, N. J.).

√*Ulam, S. Random processes and transformations. Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, pp. 264–275. Amer. Math. Soc., Providence, R. I., 1952.

An expository paper, with a number of illustrative examples, on solving mathematical problems by means of "equivalent" random processes; in other words, by playing the corresponding "game": sampling, random walk, etc. The paper is particularly authoritative since the contributions of the author had a decisive role in the development of this method of approach. *M. Loève* (Berkeley, Calif.).

Pollaczek, F. Problèmes de calcul des probabilités relatifs à des systèmes téléphoniques sans possibilité d'attente. Ann. Inst. H. Poincaré 12, 57–96 (1951).

This is a companion to two previous papers [same Ann. 11, 113–133, 135–173 (1949); these Rev. 11, 660, 672] and presents the streamlined application of the author's methods to congestion problems in telephony when waiting lines are impossible. A simple trunk group of s lines is considered with an unspecified trunk holding time distribution function, and the case where the group is preceded by a connector, whose operating time is a random variable again left general, is treated as well as the usual case where connection time is absorbed in trunk holding time. Also probabilities of order, such as the probability of serving a given call and its jth predecessor, as well as the usual probabilities of x lines busy are formulated; the former are peculiarly apt to the author's basic supposition that n calls arrive at random in an interval T. The general formulation for all probabilities considered results in a system of integral equations which are shown to have a unique solution under reasonable conditions. For the important case of exponential trunk holding time distribution, solutions are exhibited for a variety of conditions, and also for the limiting case where n/T is constant while both n and T increase. *J. Riordan* (New York, N. Y.).

Pollaczek, Félix. Application du calcul des probabilités au phénomène de blocage temporaire des lignes téléphoniques. Ann. Télécommun. 6, 49–53 (1951).

This is a resumé for engineers of previous work [C. R. Acad. Sci. Paris 226, 2045–2047 (1948); Ann. Inst. H. Poincaré 11, 135–173 (1949); these Rev. 10, 200; 11, 672; and the paper reviewed above] on loss and delay in a central with a single group of trunks preceded by a connector with operating time following a distribution left arbitrary. The trunk holding time is exponential and calls arrive at random. He first gives the probability of loss for centrals without arrangements for delay, which is simpler than that of Fortet [compare the following review], and shows that the traffic of saturation is less than the reciprocal of the average connection time, measured in units of the average holding time. For centrals with delay, and for service in order of arrival, the delay distribution function is determined for a group of

2 trunks, which is taken to simplify the complicated general formulas. *J. Riordan* (New York, N. Y.).

Fortet, Robert. Évaluation de la probabilité de perte d'un appel téléphonique, compte tenu du temps d'orientation et du groupement des lignes. Ann. Télécommun. 5, 98–113 (1950).

The author summarizes for an engineering audience his previous work [C. R. Acad. Sci. Paris 226, 159–161, 1502–1504 (1948); these Rev. 9, 361, 518] on the probability of loss in a central office having n trunk groups each of x trunks and preceded by a connector of constant operating time θ. The incoming traffic is Poisson, the trunk holding time exponential, and calls which cannot be connected are refused. An expansion of the loss probability in powers of θ, to simplify computations, is examined and rejected. In its stead, an approximation is given based on first assuming a continuous distribution function for the connector operating time and then supposing certain conditional probabilities depending on time are constants. *J. Riordan.*

Riordan, John. Telephone traffic time averages. Bell System Tech. J. 30, 1129–1144 (1951).

With the assumption of statistical equilibrium the author determines by means of generating functions the first four semi-invariants for the distribution of the mean traffic measured for a part of the busy hour. It is preassumed that the traffic sources are independent and finite or infinite in number, with an arbitrary distribution function for the holding time. When the sources are infinite he obtains the semi-invariants:

$$k_1=b,\quad k_n=b\frac{n(n-1)}{T^n}\int_0^T g(x)(T-x)x^{n-2}dx,\quad n=2, 3, 4,$$

where T is the period of observation, $g(x)=h^{-1}\int_x^\infty f(t)dt$, $h=\int_0^\infty f(t)dt$, and $f(t)$ is the probability that the holding time should be more than t time units. *A. Jensen.*

Barnard, G. A. The theory of information. J. Roy. Statist. Soc. Ser. B. 13, 46–59; discussion: 59–64 (1951). Expository paper.

Heitler, W., and Jánossy L. On the absorption of meson-producing nucleons. Proc. Phys. Soc. Sect. A. 62, 374–385 (1949).

Probabilistically speaking, the authors discuss two stochastic processes which can be described as follows. (a) A moving particle can hit other spherical nuclei. When it travels through a distance x the probability of hitting n nuclei is, by virtue of the usual assumptions, given by the Poisson distribution with parameter λx. After each hit the distance travelled through the sphere is a random variable with distribution $F(t)=t^2$ for $t\leqq 1$ and $F(t)=1$ for $t\geqq 1$. The total distance travelled through nuclear matter is then a random variable with the compound Poisson distribution $\sum e^{-\lambda x}(\lambda x)^n F^{n*}(x)/n!$, where $F^{n*}(x)$ denotes the n-fold convolution. (b) If the particle has energy E, then at a hit it may lose an amount of energy $\epsilon\leqq E$ which is a random variable with a given probability density $w(\epsilon, E)$. The probability distribution $S(E, x)$ for the energy E of a particle which has travelled through a distance x satisfies the equation

$$(*)\qquad \frac{\partial S}{\partial x}=-a(E)S+\int_E^\infty S(t, x)w(t-E, t)dt,$$

where $a(E)=\int_0^E w(\epsilon, E)d\epsilon$ is the probability of an energy loss. We have here a "purely discontinuous" stochastic process with x playing the role of time, and (*) is the so-called forward equation. The solution $S(E, x)$ is completely determined if $S(E, 0)$ is given. The authors assume the form $w(\epsilon, E)=w(\epsilon/E)E^{-1}$ and that $S(E, 0)=cE^{-\gamma}$ for $E\geqq E_0$ and $S(E, 0)=0$ for $E<E_0$. They show that, in principle, (*) can be solved by Mellin transforms. Under the special assumptions they find an expression for $S(E, x)$ valid for $E>E_0$. [The exact solution can be written down directly upon noticing that we are actually concerned with a compound Poisson process.] *W. Feller* (Princeton, N. J.).

Jánossy, L. On the absorption of a nucleon cascade. Proc. Roy. Irish Acad. Sect. A. 53, 181–188 (1950).

Jánossy, L., and Messel, H. Investigation into the higher moments of a nucleon cascade. Proc. Roy. Irish Acad. Sect. A. 54, 245–262 (1951).

The results of the preceding review are generalized to the case where a nucleus falling on another nucleus gives rise to one or more new nuclei. In the first paper it is shown that averaging leads to the same result as above. In the second paper higher moments of the distribution of the number of nuclei at given depth are calculated. It should be observed that the equation corresponding to (*) is no longer linear. *W. Feller* (Princeton, N. J.).

Messel, H., and Gardner, G. W. The solution of the Janossy G-equation. Physical Rev. (2) 84, 1256 (1951).

The authors report that they were able to solve the non-linear integro-differential equation mentioned in the last review. *W. Feller* (Princeton, N. J.).

Mathematical Statistics

Kunisawa, Kiyonori, Makabe, Hajime, and Morimura, Hidenori. Tables of confidence bands for the population distribution function. I. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, 23–44 (1951).

Tables are constructed for $\Phi(\lambda)=\sum_{k=-\infty}^{\infty}(-1)^k e^{-2k^2\lambda^2}$, $\lambda=.250(.001)2.300$. The function $\Phi(\lambda)$ is computed to seven figures after the decimal point. These tables may be applied to obtain confidence bands for the cumulative distribution function of a continuous population and are improvements over previously constructed tables by N. Smirnov. *H. Chernoff* (Urbana, Ill.).

Taguti, Gen-iti. On bias of sample mean and sample variance due to rounding or grouping. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, 9–14 (1951).

The author tabulates the bias due to rounding or grouping in the mean and the variance for critical positions of the mean with respect to the boundaries of the class within which it falls for class widths which are varying fractional and integral multiples of the standard deviation in the case the variable obeys a normal distribution law and in the case the distribution law is a simple exponential. He develops the bias in the mean and in the variance into Fourier series which R. A. Fisher first did formally [Philos. Trans. Roy. Soc. London. Ser. A. 222, 309–368 (1922)]. *C. C. Craig.*

Fréchet, Maurice. Rapport sur une enquête internationale relative à l'estimation statistique des paramètres. 25th Session of the International Statistical Institute, September, 1947, Washington, D. C., Proceedings, vol. III, pp. 363–384; discussion, pp. 384–422.

Responses and digressions, some pertinent and some rather long-winded, by sixteen statisticians from various countries to the following question: An event is observed once in one trial, what can one say about its unknown probability? *K. L. Chung* (Ithaca, N. Y.).

Nabeya, Seiji. Absolute moments in 2-dimensional normal distribution. Ann. Inst. Statist. Math., Tokyo **3**, 2–6 (1951).

The author derives expressions for the absolute moments of variables obeying a normal bivariate distribution law and lists the detailed values through order 12. *C. C. Craig.*

Moriguti, Sigeiti. Extremal properties of extreme value distributions. Ann. Math. Statistics **22**, 523–536 (1951).

The upper and lower bounds for the expectation, the coefficient of variation, and the variance of the largest member of a sample from a symmetric population are discussed. The upper bound for the expectation, the lower bound for the coefficient of variation, and the lower bound for the variance are actually achieved. The rest of the bounds are not actually achieved but approached as limits. (From the author's summary.) *H. Chernoff.*

Cansado, Enrique. On the application of the moment generating function to unrestricted random sampling. Trabajos Estadística **1**, 117–146 (1950). (Spanish. English summary)

Formulas are given relating the moments of the sample mean to the moment generating function of the population, in sampling from a finite population with and without replacement. *D. Blackwell* (Washington, D. C.).

Gini, Corrado. The means of samples. 25th Session of the International Statistical Institute, September, 1947, Washington, D. C., Proceedings, vol. III, pp. 258–271.

Consider a finite population to each element of which there corresponds a non-negative number. Various means of the means of random samples are compared with population means and each other. Let ${}^bM^p$ denote a combinatorial power mean, the bpth root of the arithmetic mean of all possible products of elements taken b at a time, each raised to the pth power. Let subscript s denote a sample mean and $\mathfrak{M}$ a mean over all possible samples of size s. Then $\mathfrak{M}^{bp}({}^bM_s{}^p) = {}_n{}^bM^p$, the population mean. Besides the well-known fact that ${}^bM^p$ increases with p, the author finds that ${}^bM_s{}^p$ varies less with p for small s and varies more with p for small b. *S. W. Nash* (Vancouver, B. C.).

Cochran, W. G. Recent developments in sampling theory in the United States. 25th Session of the International Statistical Institute, September, 1947, Washington, D. C., Proceedings, vol. III, pp. 40–66.

Expository paper with a bibliography.

Seal, K. C. On errors of estimates in various types of double sampling procedure. Sankhyā **11**, 125–144 (1951).

Expressions are obtained for estimates and their variances involved in various types of double sampling techniques pertaining to both linear and non-linear regression. The author also derives the joint distribution of regression coefficients and the expected value of a typical element in the inverse of the sample dispersion matrix for a multivariate normal population. *R. P. Peterson* (Seattle, Wash.).

Bose, Chameli. Some further results on errors in double sampling technique. Sankhyā **11**, 191–194 (1951).

The author derives error formulae for the accuracy of the estimates used in various double sampling regression schemes where the auxiliary variable x is kept constant (i) in both stages of the sampling procedure and (ii) in the first stage only. *R. P. Peterson* (Seattle, Wash.).

Narain, R. D. On sampling without replacement with varying probabilities. J. Indian Soc. Agric. Statistics **3**, 169–174 (1951).

In a subsampling system there are M primary units with known sizes. It is desired to select m of these primary units in some random fashion and to take n observations from each of these. If the primary units are to be sampled without replacements, one must consider the problem of finding those probabilities of selecting any specified set of m of these units which give rise to an unbiased estimate with small variance. The case $m=2$ is treated in this exploratory study. *H. Chernoff* (Urbana, Ill.).

Kitagawa, Tosio. Sampling from processes depending upon a continuous parameter. Mem. Fac. Sci. Kyūsyū Univ. A. **5**, 181–188 (1950).

The author obtains the mean and variance of the mean of a sample from a stochastic process with a stationary correlation function (the mean need not be stationary). It is assumed that the sampling with respect to the time parameter is independent of the sampling with respect to the point in the probability space. *J. Wolfowitz* (Ithaca, N. Y.).

√***Barankin, E. W. Conditional expectation and convex functions.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 167–169. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

For any convex function ψ on k-space, any k-dimensional chance variable f, and any chance variable τ

$$E[\psi(f)\,|\,\tau] \geqq [E(f\,|\,\tau)]$$

with probability 1. Taking expectations yields

$$E\psi(f) \geqq E\psi(E(f\,|\,\tau)),$$

a fact noted by Hodges and Lehmann [Ann. Math. Statistics **21**, 182–197 (1950); these Rev. **12**, 36].

D. Blackwell (Washington, D. C.).

Gnedenko, B. V., and Korolyuk, V. S. On the maximum discrepancy between two empirical distributions. Doklady Akad. Nauk SSSR (N.S.) **80**, 525–528 (1951). (Russian)

Let $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_m)$ be two collections of mutually independent random variables with a common continuous distribution $F(t)$. Let $F_1(t)$ and $F_2(t)$ be the corresponding empirical distributions and

$$D^+ = \sup\{F_1(t) - F_2(t)\}, \quad D = \sup|F_1(t) - F_2(t)|.$$

Smirnov gave the limiting distributions of the random variables D^+ and D as $n,\ m \to \infty$. [For new derivations cf.

Feller, Ann. Math. Statistics 19, 177–189 (1948); Doob, ibid. 20, 393–403 (1949); Kac, Proc. Nat. Acad. Sci. U. S. A. 35, 252–257 (1949); these Rev. 9, 599; 11, 43; 10, 614]. There is a considerable interest in more precise estimates for finite n, m. The authors show that when $n=m$ and c is an integer, one has the exact distributions

$$\Pr\{nD^+<c\}=1-\binom{2n}{n-c}\div\binom{2n}{n},$$

$$\Pr\{nD<c\}=\binom{2n}{n}^{-1}\sum(-1)^k\binom{2n}{n-kc},$$

the summation extending over $k=0, \pm1, \cdots, \pm[n/c]$. For the proof let $z_1, \cdots, z_{2n}$ be the collection $x_1, \cdots, y_n$ ordered according to magnitude and put $\xi_k=\pm1$ or -1 according as $z_k=x_\nu$ or $z_k=y_\nu$. The variables ξ_k determine a random walk, and the theorem becomes merely a restatement of known results concerning the ruin problem. *W. Feller.*

✱Birnbaum, Z. W. On the distribution of Kolmogorov's statistic for finite sample size. Proceedings, Seminar on Scientific Computation, November, 1949, pp. 33–36. International Business Machines Corp., New York, N. Y., 1950.

A discussion of the desirability of solving the problem mentioned in the preceding review, if necessary, by calculating machines. *W. Feller* (Princeton, N. J.).

Matthai, Abraham. Estimation of parameters from incomplete data with application to design of sample surveys. Sankhyā 11, 145–152 (1951).

Suppose that a sample from a population characterised by two measurements x and y is such that, for some of the individuals, only one or the other of the two measurements is available. Let $N=n+n_1+n_2$ be the total number of individuals in the sample where n_1 of them provide the x measurement only, n_2 provide the y measurement only and n provide both. Wilks [Ann. Math. Statistics 3, 163–195 (1932)] has shown, assuming that x and y possess a bivariate normal distribution, that the maximum likelihood estimates for the population parameters which are based on the entire sample are more accurate than those based on $n+n_1$ and $n+n_2$ individuals separately. The author extends these results to the case of more than two variables and discusses the efficiency of the estimates. Certain applications to the planning of sampling surveys on correlated variables are given. *R. P. Peterson* (Seattle, Wash.).

Lamotte, M. et Schutzenberger, M. Sur certains problèmes d'estimation dans les cas de double échantillonnage. Biometrics 7, 275–282 (1951).

Let p be distributed in [0, 1] according to a cumulative, F; and let x be an integral-valued random variable such that

$$p(x)=\int B(x|n, p)dF(p),$$

where B denotes a binomial distribution with parameters n, p. The relations (linear) between the moments of x and of p are derived. These are used to construct unbiased estimates of the moments of p and thereby a method of fitting p with polynomials both based on repeated observation of x. The analogous program is also carried through for the hypergeometric distribution. *L. J. Savage* (Paris).

Cohen, A. C., Jr. On estimating the mean and variance of singly truncated normal frequency distributions from the first three sample moments. Ann. Inst. Statist. Math., Tokyo 3, 37–44 (1951).

Estimates are found for the mean and variance of a normal population for which observations beyond a truncation point are omitted and the number of omitted observations is unknown. By using three moments the use of special tables is avoided. The estimates substitute sample moments for the moments of the truncated population. A numerical table gives the asymptotic efficiencies of the estimates as functions of the truncation point. The reader is referred to an earlier paper by the author [Ann. Math. Statistics 22, 256–265; these Rev. 12, 841] and to a summary of previous work by R. A. Fisher [Mathematical Tables, vol. 1, British Association for the Advancement of Science, 1931, pp. xxvi–xxxv]. *S. W. Nash* (Vancouver, B. C.).

Stevens, W. L. Asymptotic regression. Biometrics 7, 247–267 (1951).

The author considers the estimation of α, β, and ρ in the regression equation, $y=\alpha+\beta\rho^x$. The normal equations for the least square estimates, a, b, and r, are easily written down, but difficult to solve, so Fisher's method is used. That is, a set of preliminary estimates, a', b' and r', are improved with the help of the dispersion matrix of parameter estimates, and the procedure repeated until the least square solutions are reached. In this particular case, it is found that the improved set of estimates are functions only of r'. The author gives tables of certain functions of r' for values of r' between 0.25 and 0.75, which make the fitting of such a regression a very simple matter in the case where the x values assume 5, 6, or 7 equidistant values. Two worked examples are given, demonstrating also the calculation of the error sum of squares and tolerance limits. *P. Whittle.*

Sato, Ryoichiro. The r tests relating to the regression. Ann. Inst. Statist. Math., Tokyo 3, 45–56 (1951).

In a previous paper [same Ann. 2, 91–124 (1951); these Rev. 13, 52] the author defined "r distributions." Here the class of tests based on these distributions are extended to normal multivariate regression problems. These are again equivalent to tests based on Student's t-distribution. The method of attack is similar to that employed in the earlier paper. *D. G. Chapman* (Seattle, Wash.).

Theil, H. Distribution-free methods in the regression analysis of two variables. Statistica, Rijswijk 5, 97–117 (1951). (Dutch. English summary)

This paper gives an expository survey of the distribution-free methods of Wald, Housner and Brennan and Thiel. A discussion of the efficiency in a simple case is given.

Author's summary.

✓**✱Girshick, M. A., and Savage, L. J. Bayes and minimax estimates for quadratic loss functions.** Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 53–73. University of California Press, Berkeley and Los Angeles, 1951. $11.00.

The authors consider estimation problems in which the loss when g is the estimated value of the unknown real parameter θ is $\lambda(\theta)(g-\theta)^2$. For the case in which θ is a translation parameter, i.e. $P_\theta\{x\varepsilon S\}=P_0\{x-\theta\epsilon\,\varepsilon\,S\}$, where P_0 is a known distribution over n-space and $\epsilon=(1, 1, \cdots, 1)$, the estimate

$$u^*(x)=x_1-E_0(x_1|x-\bar{x}),$$

where $x=(x_1, \cdots, x_n)$ and $\bar{x}=\sum x_i/n$, is unbiased, has constant variance σ^2, increases by h when all observations increase by h, and no other estimate u exists with $\sup_u \sigma^2(u)<\sigma^2$. For the case in which θ is the parameter of an exponential family, i.e. the density of x is $\beta(\tau)$ with respect to a measure ψ, where $\theta=E(x|\tau)$, then if $-\infty<\tau<\infty$, x is an admissible minimax estimate of θ for $\lambda(\theta)=1/\sigma^2(x|\tau)$. Moreover, the conditional risk after n observations is a function of n only, so that, if the cost of n observations is a function of n only, the minimax sampling plan is a fixed sample size.

D. Blackwell (Washington, D. C.).

Wald, Abraham. Sequential analysis. 25th Session of the International Statistical Institute, September, 1947, Washington, D. C., Proceedings, vol. III, pp. 67–73; discussion, pp. 74–80.

Expository paper.

David, F. N., and Johnson, N. L. The sensitivity of analysis of variance tests with respect to random variation between groups. Trabajos Estadística **2**, 179–188 (1951). (English. Spanish summary)

This is a continuation of a previous paper by the same authors [Biometrica **38**, 43–57 (1951); these Rev. **13**, 53]. Given a one-way classification with underlying model $x_{ti}=A+u_t+z_{ti}$, where A is a constant, the z's and u's are mutually independent variables with zero means, the z's belonging to the same group, as well as the u's, having identical distributions. Let

$$S_1=\sum_{t=1}^{s} n_t(\bar{x}_{t.}-\bar{x}_{..})^2, \quad S_2=\sum_{t=1}^{s}\sum_{i=1}^{n_t}(x_{ti}-\bar{x}_{t.})^2,$$

$$a=-\frac{s-1}{N-s}F_{s-1,N-s}(\alpha).$$

The authors evaluate the first four moments of S_1+aS_2 in terms of the cumulants of the u_t and z_{ti}. By fitting suitable frequency curves to these moments, $p\{S_1+aS_2>0\}$ can be evaluated, giving an indication of the sensitivity of the F-test if some of the usual assumptions of the analysis of variance are not satisfied. Various special cases are considered.

G. E. Noether (Boston, Mass.).

***Kempthorne, Oscar. The design and analysis of experiments.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1952. xix+631 pp. $8.50.

This is an almost complete account of the analysis of variance and of experimental designs to which the analysis of variance is applicable. The general background is given in the first four chapters. The fifth chapter discusses the likelihood ratio principle and the testing of linear hypotheses. This is applied in the following chapters to the analysis of r-way classification designs, randomized block designs, including a discussion of the analysis of covariance, and orthogonal Latin squares. Chapter 12 discusses the power-function of the analysis of variance test and the use of Tang's tables in finding the power function of a given test as well in solving the converse problem of designing the experiment so as to achieve a prescribed power with respect to given alternatives under certain assumptions. The next 6 chapters deal with factorial experiments in complete replications and with the theory of confounding and partial confounding in such experiments. The concept of main effects and interactions are first introduced in an intuitive manner and later formally defined as the means of the solutions of certain linear equations in a finite field. Particular attention is given to the 2^n design and the 3^n design. There follows a discussion of split plot designs and two chapters on the theory of fractional replication. The next 6 chapters are devoted to a thorough discussion of various designs useful in varietal trials: lattices, incomplete block designs, rectangular lattices, balanced and partially balanced incomplete block designs. The last two chapters are devoted to the design and analysis of groups of experiments and to experiments applying treatments in sequence.

The book contains detailed accounts of each of these experimental techniques. Numerous examples illustrate the various situations in which these techniques can be employed. The discussion of the basic theory is adequate but it is the reviewer's impression that the basic principles are somewhat lost in the long and detailed discussion of special cases and examples. This may, however, be just the prejudice of a mathematician who likes to obtain his information in the form of a sequence of precise definitions and theorems rather than through a discussion of special cases and examples.

H. B. Mann (Columbus, Ohio).

***Tukey, John W. Standard methods of analyzing data.** Proceedings, Computation Seminar, December 1949, pp. 95–112. International Business Machines Corp., New York, N. Y., 1951.

This paper is an exposition of the role of mathematical models in statistics and of the interpretation of formulae which frequently arise in regression and analysis of variance. Various cases of increasing complexity are treated.

H. Chernoff (Urbana, Ill.).

TOPOLOGY

Maunsell, F. G. A note on Tutte's paper "The factorization of linear graphs." J. London Math. Soc. **27**, 127–128 (1952).

This note refers to a paper by the reviewer in which a necessary and sufficient condition for a given graph to have a 1-factor is derived [same J. **22**, 107–111 (1947); these Rev. **9**, 297]. Part of the reviewer's proof depends on the properties of skew-symmetric determinants. The author replaces this part by an argument using only the elementary properties of graphs. It should be remarked that the more general theory of Belck also makes no use of determinants [J. Reine Angew. Math. **188**, 228–252 (1950); these Rev. **12**, 730].

W. T. Tutte (Toronto, Ont.).

Dirac, G. A. A property of 4-chromatic graphs and some remarks on critical graphs. J. London Math. Soc. **27**, 85–92 (1952).

This paper is concerned with colourings of the vertices of a graph in k colours so that no two of the same colour are joined by an edge. The graph is called k-chromatic if k is the least integer for which this is possible. A k-chromatic graph is called critical if the suppression of an arbitrary vertex, with its incident edges, makes the graph $(k-1)$-chromatic. A complete n-graph is a graph of n vertices, each pair of vertices being joined by just one edge. It has been conjectured that a k-chromatic graph always contains a complete k-graph or a graph which can be derived from a

complete k-graph by subdividing the edges. The author prove this for $k \leqq 4$. In the remainder of his paper the author obtains some relations between the number of vertices and the number of edges of a critical k-chromatic graph.

W. T. Tutte (Toronto, Ont.).

Jones, F. Burton. Certain homogeneous unicoherent indecomposable continua. Proc. Amer. Math. Soc. 2, 855–859 (1951).

Recent papers of R. H. Bing [Duke Math. J. 15, 729–742 (1948); these Rev. 10, 261] and E. E. Moise [Trans. Amer. Math. Soc. 64, 57–58 (1949); these Rev. 11, 382] have revived interest in the study of bounded, nondegenerate, homogeneous plane continua which do not separate the plane. The papers cited above establish the existence of such sets, while the present paper proves that every such continuum is necessarily indecomposable. This result follows from the more general theorem, here proved by the author, that every homogeneous, hereditarily unicoherent, compact metric continuum is indecomposable. The proof is based upon a lemma which seems to the reviewer to have interest in its own right. Let M be a continuum and x a point of M. Let U_x denote the set of all points z of M such that M is aposyndetic at z with respect to x. Evidently, U_x is an open subset of M. The lemma follows: If the compact metric continuum M is homogeneous and x, y are distinct points of M, then U_y is not a proper subset of U_x. The proof of the lemma makes use of transfinite induction. *D. W. Hall.*

Butcher, G. H. An extension of the sum theorem of dimension theory. Duke Math. J. 18, 859–874 (1951).

This paper introduces the class of K-separable spaces. A metric space X is said to be ω-void provided it contains no pair of distinct points x, y with $\rho(x, y)$ less than ω. If $X=p$ is a single point, then X is 1-void. A space X is K-separable provided the following two conditions are satisfied: (i) X is a metric space. (ii) For each integer i there exists a subset P_i of X and a positive number ω_i such that (1) P_i is ω_i-void and (2) for each point x of X there exists a positive number $g(x)$ such that x lies in the closure of the union of all P_i for which $g(x)$ does not exceed ω_i.

It is proved that the property of being K-separable is hereditary, additive, and topological. Examples of spaces which are not K-separable are given. Urysohn has given [Fund. Math. 9, 119–121 (1927)] an example of a metric space having the property that it contains no non-empty open set which is separable. This space is shown to be K-separable. The equivalence of the Urysohn-Menger dimension function $d_1(x)$ and the dimension function $d_2(x)$ (separation of disjoint closed sets) for K-separable spaces is proved together with other equivalences and theorems. Finally, the sum and decomposition theorems for n-dimensional sets are established for K-separable spaces.

D. W. Hall (College Park, Md.).

Morita, Kiichi. On the dimension of normal spaces. I. Jap. J. Math. 20, 5–36 (1950).

The author extends to normal spaces (and generalizes in other ways) some of the familiar dimension-theoretic results of separable metric spaces. The covering definition is used. Several modifications of the sum theorem are given as well as a generalization of a theorem of Eilenberg and Otto [Fund. Math. 31, 149–153 (1938)]. Unfortunately, much of the material is not now novel since the author (at the time the paper was written) was not familiar with the results of Alexandroff [Proc. Roy. Soc. London. Ser. A. 189, 11–39 (1947); these Rev. 9, 52], Dowker [Amer. J. Math. 69, 200–242 (1947); these Rev. 8, 594] and Hemmingsen [Duke Math. J. 13, 495–504 (1946); these Rev. 8, 334]. The reception date of the paper is Aug., 1947. Theorem 6.4 could have been stated more carefully. In general, the author's results are precise and the proofs straightforward and self-contained. There is some overlap (in method and result) with Kuratowski [Topologie, v. II, Monografie Matematyczne, vol. 21, Warszawa-Wroclaw, 1950; these Rev. 12, 517].

A. D. Wallace (New Orleans, La.).

Vilenkin, N. Ya. On the determination of the dimension of a compact metric space by means of the ring of continuous functions on it. Uspehi Matem. Nauk (N.S.) 6, no. 5(45), 160–161 (1951). (Russian)

Let X be a compact metric space and let $C(X)$ be the ring of all real-valued continuous functions on X. Then the following two conditions are necessary and sufficient for X to have dimension n. (1) For every $n+1$ functions $\varphi_1, \varphi_2, \cdots, \varphi_{n+1} \varepsilon C(X)$ and every $\epsilon>0$, there exist functions $\xi_1, \xi_2, \cdots, \xi_{n+1} \varepsilon C(X)$ such that $\|\varphi_i-\xi_i\|<\epsilon\ (i=1, 2, \cdots, n+1)$ and the functions $\xi_1, \xi_2, \cdots, \xi_{n+1}$ are contained in no single maximal ideal of $C(X)$; (2) there exist $f_1, f_2, \cdots, f_n \varepsilon C(X)$ and $\delta>0$ such that if $g_1, g_2, \cdots, g_n \varepsilon C(X)$ and $\|f_i-g_i\|<\delta$ $(i=1, 2, \cdots, n)$, then $g_1, g_2, \cdots, g_n$ lie in a proper closed ideal in $C(X)$. This ring-theoretic characterization of dimension is to be compared with that given by Katětov [Časopis Pěst. Mat. Fys. 75, 1–16 (1950); these Rev. 12, 119].

E. Hewitt (Seattle, Wash.).

Obreanu, Filip. On a problem of Aleksandrov and Urysohn. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 101–108 (1950). (Romanian. Russian and French summaries)

The author presents a new proof of a conjecture of Aleksandrov and Uryson, first established by M. H. Stone [Trans. Amer. Math. Soc. 41, 375–481 (1937), p. 435]. In essence, the proof is the same as Stone's. *E. Hewitt.*

Mackina, R. Yu. A universal continuous mapping of Hilbert space. Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, 533–544 (1951). (Russian)

Let H and H' be separable Hilbert spaces. There exists a continuous mapping of H into H' such that for every analytic subset B of H' and every open subset G of H, there exists a closed subset A of G such that $f(A)$ is homeomorphic to B. *E. Hewitt* (Seattle, Wash.).

Vincze, St., und Szüsz, P. Beweis eines Abbildungssatzes von Béla Sz.-Nagy. Acta Sci. Math. Szeged 14, 96–100 (1951).

If M and N are subsets of a metric space and f is a map of M into N, then f will be called Lipschitzian (authors' term: dehnungsbeschränkt) if $\sup_{x, y \varepsilon M} \rho(f(x), f(y))/\rho(x, y)<\infty$. Now suppose M is the unit cell $\{\|x\| \leqq 1\}$ in E_n, N is a bounded closed convex body whose interior contains 0, and f is the natural radial map of M onto N. The authors prove that both f and f^{-1} are Lipschitzian. *V. L. Klee, Jr.*

Floyd, E. E. Examples of fixed point sets of periodic maps. Ann. of Math. (2) 55, 167–171 (1952).

Let L be the fixed-point set of a homeomorphism of prime period p operating in a space X. It is known that if X is a compact finite-dimensional homology sphere over J_p (integers mod p) so is L. The question whether this is true for

coefficient groups which are not dependent on p is of interest, since an affirmative answer would lead quickly to knowledge about the fixed points of transformations of non-prime periods. The author shows, however, that for every nontrivial abelian group G there exists a prime p, a finite complex K and a simplicial map of K onto itself of period p such that K is a homology sphere over G but its subcomplex of fixed points is not. (There is perhaps some significance in the fact that the K's exhibited are not homology spheres locally.) *P. A. Smith* (New York, N. Y.).

Moise, Edwin E. Affine structures in 3-manifolds. II. Positional properties of 2-spheres. Ann. of Math. (2) 55, 172–176 (1952).

This is the second in a series of papers [for part I see same Ann. 54, 506–533 (1951); these Rev. 13, 484] on topological 3-manifolds. J. W. Alexander [Proc. Nat. Acad. Sci. U. S. A. 10, 6–8 (1924)] proved the Jordan-Schoenflies theorem for a polyhedral 2-sphere in euclidean 3-space E^3. Moise strengthens this result by showing that, if S and S' are two such polyhedra inside a cube, then there exists a piecewise linear self-homeomorphism of E^3 which is the identity outside the cube and maps S onto S'. His proof is more elementary and direct than Alexander's. A second theorem asserts that the complement of a set K in E^3 is homeomorphic to the complement of the unit sphere if there exists an open neighborhood U of K and a homeomorphic mapping f such that $f(U)\subset E^3$ and $f(K)$ is a polyhedral sphere. *S. S. Cairns* (Urbana, Ill.).

***Eckmann, Beno. Continu et discontinu.** Congrès International de Philosophie des Sciences, Paris, 1949, vol. III. Philosophie Mathématique, Mécanique, pp. 67–74. Actualités Sci. Ind., no. 1137. Hermann & Cie., Paris, 1951.

***Eckmann, Beno. Complex-analytic manifolds.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 420–427. Amer. Math. Soc., Providence, R. I., 1952.

A survey of methods used and results obtained in the theory of complex manifolds. The first part describes the results obtained in the theory of almost complex manifolds by Hopf, Ehresmann, and others, with particular reference to spheres S^{2k}. The second part describes some of the results obtained by numerous authors concerning the topological properties of complex manifolds which carry Kählerian metrics. *W. V. D. Hodge* (Cambridge, England).

***Ehresmann, Charles. Sur les variétés presque complexes.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 412–419. Amer. Math. Soc., Providence, R. I., 1952.

This lecture describes mainly the contributions of the author and of Wen-tsün Wu. It starts with the idea of subordinated fiber bundle (reduction of the structure group G to a subgroup G', relation to the associated bundle with fiber G/G'). If the fiber is real $2n$-space R^{2n}, one defines, by singling out certain subgroups of the group $G=L_{2n}$ = general linear group, a series of subordinated structures: oriented-real, almost complex, almost quaternionic, Riemannian, almost hermitian, almost quaternion-hermitian. For almost complex structures various points are discussed, e.g., the equivalence with the existence of a 2-form Ω with $\Omega^n\neq 0$; symplectic and almost Kähler manifolds are defined. The space $\Gamma_n=O^+{}_{2n}/O'_n$ = connected orthogonal group modulo unitary group is studied. Since $\pi_2(\Gamma_n)$ is infinite cyclic, one gets a primary obstruction W^3 against existence of an almost complex structure; W^3 is identical with the Stiefel-Whitney 3-class. Results of Wu give connections between the Stiefel-Whitney- and Pontryagin-classes, and the Chern classes of a subordinated almost complex structure, and a construction of all almost complex structures on a V_4. Properties of real and almost complex submanifolds of almost complex manifolds are studied; e.g., for a real V_n in complex projective space $P_n(C)$ all cycles of dimension $4k+2$ bound in $P_n(C)$. Equivalence of structures is discussed with E. Cartan's methods, an integrability condition for an almost complex structure to be true-complex is set up, in terms of differential forms. There is some overlap with Ehresmann, Topologie algébrique, pp. 3–15, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12, Paris, 1949; these Rev. 11, 678. *H. Samelson.*

Hirzebruch, Friedrich. Über eine Klasse von einfach-zusammenhängenden komplexen Mannigfaltigkeiten. Math. Ann. 124, 77–86 (1951).

Two complex manifolds are analytically equivalent if they can be topologically and analytically mapped onto one another. This paper makes a contribution to the general problem of determining all analytic equivalence classes among a class of topologically equivalent manifolds. Attention is confined to certain fiber bundles Σ_n $(n=0, 1, 2, \cdots)$ with S^2 for fiber and S^2 for base space. For n even, Σ_n is homeomorphic to $S^2\times S^2$, and for n odd to $P^{(2)}+P^{(2)}$ (the topological sum of two complex projective planes). For $n\neq m$, Σ_n and Σ_m are complex manifolds belonging to different analytic equivalence classes. As algebraic surfaces, all the spaces Σ_n are birationally equivalent. *S. S. Cairns.*

***Hopf, Heinz. Die n-dimensionalen Sphären und projektiven Räume in der Topologie.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 193–202. Amer. Math. Soc., Providence, R. I., 1952.

This paper is a survey of results and problems concerning the n-sphere and the real projective n-space. Among the topics discussed are the existence of vector fields on spheres, the existence of hypercomplex systems over the real field, the existence of complex and almost complex structures on spheres, the existence of essential mappings of spheres on spheres, and the imbedding of real projective spaces in euclidean spaces. In each case the known results are presented along with the questions still open. Some of these questions have been solved or partially solved since the report was given; in particular, new results have been obtained concerning the existence of vector fields on spheres [Steenrod and Whitehead, Proc. Nat. Acad. Sci. U. S. A. 37, 58–63 (1951); these Rev. 12, 847], the existence of almost complex structures on spheres [Borel and Serre, C. R. Acad. Sci. Paris 233, 680–682 (1951); these Rev. 13, 319], and the mappings of spheres onto spheres [see the preceding review]. *E. H. Spanier* (Chicago, Ill.).

Serre, Jean-Pierre. Homologie singulière des espaces fibrés. Applications. Ann. of Math. (2) 54, 425–505 (1951).

In this paper the author gives a complete exposition (with proofs) of some results announced previously [C. R. Acad. Sci. Paris 231, 1408–1410 (1950); 232, 31–33, 142–144 (1951); these Rev. 12, 520, 521]. The paper is divided into five chapters. The first chapter gives a very readable account of the purely algebraic aspects of the spectral sequence associated with a differential filtered group or ring. Also

included is an abstract theory of the "transgression" and "suspension". In chapter II the properties of the spectral (singular) homology sequence of a fibre space are established. For this purpose, it is necessary to use a new definition of singular homology groups which uses singular "cubes" instead of the customary singular simplexes. A brief discussion of this new method of defining singular homology groups is given, and the reader is referred to a forthcoming paper of Eilenberg and MacLane for more details. A point which requires particular care here is the question of which singular cubes shall be considered "degenerate". A filtration is defined on the group of all cubical singular chains of a fibre space, and the author shows that the first term of the spectral sequence which then arises is isomorphic to the group of chains of the base space with coefficients in the homology groups of the fibre.

Chapters III, IV, V, and VI give various applications of the theory which is developed in the first two chapters. For a resumé of the principal results here, the reader is referred to the reviews of the three notes cited above. There is also a short appendix in which the following two results on the relation between the homology groups of a base space X and its universal covering space T are proved. Let k be a commutative field and assume that Π, the fundamental group of X, operates trivially on the homology groups $H_i(T, k)$ for all dimensions $i \geqq 0$. Then: (1) If Π is infinite cyclic, $H_i(X, k)$ is isomorphic to the direct sum of $H_i(T, k)$ and $H_{i-1}(T, k)$. (2) If Π is a finite group whose order is not divisible by the characteristic of k, then $H_i(X, k)$ is isomorphic to $H_i(T, k)$. The author makes essential use of these two results in his proofs. *W. S. Massey.*

Wada, Hidekazu. On the structure of a sphere bundle. Tôhoku Math. J. (2) **3**, 136–139 (1951).

In this note the following two theorems are proved: (1) Let p: $M \rightarrow S$ be a fibre mapping with base space S an m-sphere, and the fibre an n-sphere. If the homotopy groups $\pi_q(M)$ vanish for $q \leqq m$, then $n = m-1$. (2) Let M be a $(2m-1)$-dimensional orientable manifold which is an $(m-1)$-sphere bundle over an m-sphere S. If the projection π: $M \rightarrow S$ is algebraically inessential, then its Hopf invariant must be ± 1. *W. S. Massey* (Providence, R. I.).

✓***Eilenberg, Samuel. Homotopy groups and algebraic homology theories.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 350–353. Amer. Math. Soc., Providence, R. I., 1952.

L'auteur annonce deux théorèmes (th. 2 et 3) concernant les problèmes de classification et d'extension d'applications d'un complexe K dans un espace X tel que $\pi_i(X) = 0$ pour $i < n$ $(n > 1)$; ils généralisent les résultats classiques de Steenrod valables pour le cas où $X = S^n$ [Ann. of Math. (2) **48**, 290–320 (1947); ces Rev. **9**, 154], et englobent ceux de Whitney [ibid. **50**, 270–284 (1949); ces Rev. **11**, 531] et de Postnikov [Doklady Akad. Nauk SSSR **67**, 427–430 (1949); ces Rev. **11**, 451].

L'auteur rappelle d'abord la définition de l'obstruction $k_n^{q+1} \varepsilon H^{q+1}(\pi_n, n; \pi_q)$ d'un espace X connexe par arcs, tel que $\pi_i(X) = 0$ pour $i < n$ et $n < i < q$, $\pi_n(X) = \pi_n$, $\pi_q(X) = \pi_q$. L'invariant k_n^{q+1} sert à calculer les "secondes obstructions": si f et g appliquent $K^n \cup L$ (L: sous-complexe de K) dans X, sont prolongeables en $K^{n+1} \cup L \rightarrow X$, et coïncident sur L, l'homomorphisme $(f-g)^*$: $H^{q+1}(\pi_n, n; \pi_q) \rightarrow H^{q+1}(K, L; \pi_q)$ transforme k_n^{q+1} dans la différence des "secondes obstructions" $z^{q+1}(f)$ et $z^{q+1}(g) \varepsilon H^{q+1}(K, L; \pi_q)$. Ce théorème (th. 1) est la clef des th. 2 et 3, relatifs au cas où $q = n+1$. D'abord, pour $n \geqq 3$,

$$k_n^{n+2} \varepsilon H^{n+2}(\pi_n, n; \pi_{n+1}) \approx \mathrm{Hom}\,(\pi_n/2\pi_n, \pi_{n+1})$$

s'identifie à l'homomorphisme η: $\pi_n \rightarrow \pi_{n+1}$ de J. H. C. Whitehead, défini à partir d'une application essentielle $S^{n+1} \rightarrow S^n$. Pour $n = 2$, $H^4(\pi_2, 2; \pi_3) \approx \mathrm{Hom}\,(\Gamma(\pi_2), \pi_3)$, où $\Gamma(A)$ désigne le groupe abélien que Whitehead associe à tout groupe abélien A; Hom $\Gamma((\pi_2), \pi_3)$ s'identifie au groupe des applications "quadratiques" t: $\pi_2 \rightarrow \pi_3$, c'est-à-dire telles que $t(-x) = t(x)$ et que l'application $(x, y) \rightarrow t(x+y) - t(x) - t(y)$ de $\pi_2 \times \pi_2$ dans π_3 soit bilinéaire. L'élément $k_2^4 \varepsilon H^4(\pi_2, 2; \pi_3)$ s'identifie à l'application quadratique η: $\pi_2 \rightarrow \pi_3$ définie par une application $S^3 \rightarrow S^2$ correspondant au générateur de $\pi_3(S^2)$.

Les th. 2 et 3 se rapportent au cas où $\pi_i(X) = 0$ pour $i < n$, $\pi_n(X)$ étant de type fini. Théorème 2: si deux applications f, g satisfont aux conditions du th. 1, la différence $z^{n+2}(f) - z^{n+2}(g) \varepsilon H^{n+2}(K, L; \pi_{n+1})$ est égale (si $n \geqq 3$) au carré de Steenrod $\mathrm{Sq}^2(\lambda)$, $\lambda \varepsilon H^n(K, L; \pi_n)$ étant l'image par $(f-g)^*$ de la classe fondamentale de $H^n(X; \pi_n)$, et Sq^2 étant pris relativement à l'application η: $\pi_n \rightarrow \pi_{n+1}$; le cas $n = 2$ est plus compliqué et fait intervenir le "carré de Pontrjagin" de λ, pris relativement à l'application η: $\pi_2 \rightarrow \pi_3$. Le th. 3 traite de la déviation $d^{n+1}(f, g) \varepsilon H^{n+1}(K, L; \pi_{n+1})$ de 2 applications f, g de K^{n+1} dans X, qui coïncident sur $K^n \cup L$; il dit à quelle condition doit satisfaire $d^{n+1}(f, g)$ pour que f et g soient L-homotopes: si $n \geqq 3$, $d^{n+1}(f, g)$ doit être le Sq^2 d'un élément de $H^{n-1}(K, L; \pi_n)$; si $n = 2$, la condition, plus compliquée, fait intervenir le "carré de Postnikov" d'un élément de $H^1(K, L; \pi_2)$. *H. Cartan* (Paris).

✓***Whitehead, J. H. C. Algebraic homotopy theory.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 354–357. Amer. Math. Soc., Providence, R. I., 1952.

Exposé de résultats principalement contenus dans: S. MacLane and J. H. C. Whitehead [Proc. Nat. Acad. Sci. U. S. A. **36**, 41–48 (1950); ces Rev. **11**, 450], resp. dans J. H. C. Whitehead [Ann. of Math. **52**, 51–110 (1950); ces Rev. **12**, 43]; il s'agit du 3-type d'un CW-complexe, resp. du 4-type d'un CW-complexe simplement connexe. Deux CW-complexes K, K' ont le même n-type s'il existe des applications des n-squelettes φ: $K^n \rightarrow K'^n$, φ': $K'^n \rightarrow K^n$ telles que les restrictions de $\varphi'\varphi$ à K^{n-1} et de $\varphi\varphi'$ à K'^{n-1} soient homotopes à l'identité (dans K^n, resp. K'^n). Le 3-type de K est caractérisé par les groupes $\pi_1(K)$, $\pi_2(K)$, les opérations de π_1 dans π_2, et l'invariant d'Eilenberg-MacLane $\mathbf{k} \varepsilon H^3(\pi_1, 1, \pi_2)$; d'où la notion abstraite de "3-type algébrique". Pour tout 3-type algébrique il existe un complexe K qui lui donne naissance; si K et K' sont deux complexes, pour tout "homomorphisme" du 3-type de K dans le 3-type de K' il existe une application $K \rightarrow K'$ qui lui donne naissance.

Le 4-type d'un K simplement connexe est caractérisé par les groupes (abéliens) $\pi_2(K)$, $\pi_3(K)$ et un élément $\mathbf{i} \varepsilon \mathrm{Hom}\,(\Gamma(\pi_2), \pi_3)$, où $\Gamma(\pi_2)$ désigne un groupe abélien déduit de π_2 au moyen d'une construction par générateurs et relations. En fait, $\Gamma(\pi_2)$ n'est autre que le groupe d'Eilenberg-MacLane $H_4(\pi_2, 2)$, et $\mathrm{Hom}\,(\Gamma(\pi_2), \pi_3) \approx H^4(\pi_2, 2, \pi_3)$; $\mathbf{i}$ n'est autre que l'invariant $\mathbf{k} \varepsilon H^4(\pi_2, 2, \pi_3)$ défini par Eilenberg et MacLane [Ann. of Math. **51**, 514–533 (1950); ces Rev. **11**, 735)]. Il n'est pas certain que si deux applications $K^n \rightarrow K'^n$ définissent le même homomorphisme du 4-type de K dans celui de K', leurs restrictions à K^{n-1} soient homotopes dans K'^n. *H. Cartan* (Paris).

The necessary qualifiers which the last sentence of the review calls for regarding the spheres to which the given formula applies were, by mistake, omitted from the German summary and are to be found in the Croatian text of the paper.

GEOMETRY

Thébault, Victor. Sur la géométrie du triangle. Ann. Soc. Sci. Bruxelles. Sér. I. 65, 79–86 (1951).

Devidé, Vladimir. Verallgemeinerung zweier planimetrischen Theoreme auf den n-dimensionalen Raum. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 6, 145–154 (1951). (Serbo-Croatian. German summary)

(a) If the altitudes AA', BB', CC' of a triangle $(T)=ABC$ meet the circumcircle of (T) again in the points A'', B'', C'', then

$$\frac{AA''}{AA'}+\frac{BB''}{BB'}+\frac{CC''}{CC'}=4.$$

The author proves analytically that the analogous proposition is valid for a simplex S_n determined by $n+1$ points in an n-dimensional Euclidean space, the value of the sum being equal to $2n$.

(b) If r_2 is the radius of the inscribed circle of a triangle, and $r_{2,i}$ $(i=1, 2, 3)$ the radii of its escribed circles, we have $\sum_{i=1}^{3} r_{2,i}=1/r_2$. For the simplex S_n the author obtains the formula $\sum_{i=1}^{n} r_{n,i}=(n-1)/r_n$. However, this formula can, at best, be valid if only spheres of certain types of contact with the faces of the simplex are considered, as is seen in the case of $n=3$ [see, for inst., the reviewer's Modern pure solid geometry, Macmillan, New York, 1935, pp. 82–84].

N. A. Court (Norman, Okla.).

Bishara, S., and Amin, A. Y. On two triangles whose apolar locus and apolar envelope are apolar. Proc. Math. Phys. Soc. Egypt 4, no. 2, 65–69 (1951). (English. Arabic summary)

Kasner, Edward, and Harrison, Irene. The trisection of horn angles. Scripta Math. 17, 231–235 (1951).

Narayana Moorty, T. Generalisation of Tucker's system of circles. Math. Student 19, 30–32 (1951).

Rodeja F., E. G.-. Symbolic expressions of the equations of parabolas passing through four points. Revista Mat. Hisp.-Amer. (4) 11, 257–265 (1951). (Spanish)

Lombardo-Radice, Lucio. Assiomi algebrici e postulati geometrici. Archimede 3, 177–188 (1951).

This is a study based on work of the reviewer [Trans. Amer. Math. Soc. 54, 229–277 (1943); these Rev. 5, 72] and that of B. I. Argunov [Mat. Sbornik N.S. 26(68) 425–456 (1950); these Rev. 12, 525]. A projective plane can be coordinatized by a ternary ring in which $y=x\cdot mob$ is the equation of a line through $(0, b)$ and a point (m) at infinity associated with its "slope" m. Each configuration has a local equivalent, the relation in the ring given by a single configuration. A configuration theorem valid for every placement of a configuration in the plane is equivalent to laws for the ternary operation $x\cdot mob$. *Marshall Hall.*

da Silva Dias, C. L. Complement to the work of Cohn-Vossen: The collineations of complex projective space of n dimensions. Bol. Soc. Mat. São Paulo 2, no. 2 (1947), 37–41 (1949). (Portuguese)

Cohn-Vossen [Math. Ann. 115, 80–86 (1937)] a étudié le problème de la décomposition d'un espace projectif R soumis à une homographie donnée, en envisageant le cas où l'homographie admet un nombre fini de points unis, et celui où les points unis constituent un espace linéaire unique. Van der Waerden [voir p. 83 de l'oeuvre citée] a observé, au sujet des résultats obtenus par Cohn-Vossen, que tout espace R peut être univoquement décomposé en espaces partiels unis soumis à l'hypothèse du deuxième des deux cas ci-dessus envisagés par Cohn-Vossen. Van der Waerden fournit de son assertion une démonstration analytique. L'auteur en donne une nouvelle, plus conforme à la méthode utilisée par Cohn-Vossen dans son travail, et ne faisant appel à aucune considération de systèmes de coordonnées.

P. Vincensini (Marseille).

Hoffman, A. J. Chains in the projective line. Duke Math. J. 18, 827–830 (1951).

Ist E ein quadratischer Erweiterungskörper des Grundkörpers F, n eine projektive Gerade über E, m eine projektive Gerade über F, so wird als Kette in n der Bild von m verstanden bei einer projektiven Transformation von n. Ist insbesondere F der Körper der reellen Zahlen, E der Körper der komplexen Zahlen, so sind die Ketten die Kreise einer reellen Inversionsgeometrie. Mit Hilfe genauerer Strukturuntersuchungen des Erweiterungskörpers, die mehrere Fallunterscheidungen erforderlich machen, wird für Ketten der Satz von Miquel und das Theorem von Staudt bewiesen, d.h. die Sätze: 1) Wenn sich aus vier Paaren verschiedener Punkte auf fünf Weisen zwei Paare so auswählen lassen, dass diese vier Punkte in einer Kette enthalten sind, so ist das auch bei der sechsten Auswahl der Fall. 2) Jede eineindeutige Abbildung der Punkte von n auf sich, die Ketten erhält, lässt sich zusammensetzen aus einem Automorphismus von E, der F in sich überführt, und einer projektiven Transformation von n. *R. Moufang* (Frankfurt a.M.).

Ladopoulos, Panaïotis. Sur la métrique des courbes algébriques. C. R. Acad. Sci. Paris 233, 1417–1418 (1951).

Let C_I, C_J denote the two curves (isotropics) of a pencil of algebraic plane curves of order n that pass through the circular points at infinity I, J, respectively. If $C^{(1)}$, $C^{(2)}$ are curves of the pencil, the author defines their angle by $\omega_{12}=(1/2i)(C^{(1)}, C^{(2)}; C_I, C_J) \bmod \pi$, where the second set of parentheses denotes the cross ratio of the four curves indicated. It is shown that this angle equals (1) the sum of the n angles that each asymptote of the one curve makes with each asymptote of the other (each asymptote being taken only once) and (2) the difference of the orientations of the asymptotes with respect to the same axis.

L. M. Blumenthal (Los Angeles, Calif.).

Convex Domains, Extremal Problems

Hadwiger, H., und Glur, P. Zerlegungsgleichheit ebener Polygone. Elemente der Math. 6, 97–106 (1951).

Let F be the group of all translations and point-reflections in the euclidean plane [the vector plane would serve just as well]. Two polygons A and B are said to be F congruent if there exists a transformation f in F throwing A onto B. Two polygons A and B are said to be D congruent if there exist finite nonoverlapping decompositions A_ν of A and B_ν of B such that A_ν is F congruent to B_ν for all ν. It is easily shown that F and D congruence are equivalence relations on poly-

gons. The following theorem Δ is proved: two proper polygons are D-congruent if and only if they have the same area. Since the set Φ of all bounded additive polygon functionals invariant under F congruence is generated by any area functional ϕ_P relative to a standard proper parallelogram P, theorem Δ may be rephrased as follows: two proper polygons A and B are D-congruent if and only if $\phi A=\phi B$ for every ϕ in Φ.

Let F_+ be the group of all translations. Define F_+ and D_+ congruence; they are equivalence relations. The set Φ_+ of all bounded additive polygon functionals invariant under F_+ congruence is generated by the area functionals ϕ_P together with certain other functionals ϕ_p defined for each proper vector p, ϕ_p of a polygon A being the total algebraic length relative to p of those vector bounding segments of A which are parallel to p [Hadwiger, Publ. Math. Debrecen 1, 104–108 (1949); these Rev. 12, 124]. The following theorem Δ_+ is proved: two proper polygons are D_+ congruent if and only if $\phi A=\phi B$ for every ϕ in Φ_+.

Theorems Δ and Δ_+ obviously imply the additive cancellation law (on proper polygons) for D and D_+ congruence. Actually the additive cancellation law for D_+ congruence is established independently and used in the proof of Δ_+.

W. Gustin (Bloomington, Ind.).

Tarski, Alfred. On the degree of equivalence of polygons. Riveon Lematematika 5, 32–38 (1951). (Hebrew)

This is the translation of a paper which appeared in Młody Matematyk 1, 37–44 (1931). Two polygons, U and V, are equivalent if they may be decomposed into disjoint polygons, $A_1, \cdots, A_n$, and $B_1, \cdots, B_n$, respectively, with A_i congruent to B_i for each i. The degree of equivalence of U and V is the smallest number n of polygons in such decompositions. Some properties of this concept are derived by elementary means.

M. Jerison (Lafayette, Ind.).

Goldberg, Michael. Rotors in spherical polygons. J. Math. Physics 30, 235–244 (1952).

The author considers the analogue for regular spherical polygons of his circular-arc rotors in regular plane polygons [Amer. Math. Monthly 55, 393–402 (1948); these Rev. 10, 205]. In spite of their superficial resemblance to the plane rotors, these spherical curves cannot consist of arcs of small circles, and the author regrets his inability to find a general equation analogous to that given for a plane rotor by Meissner [Vierteljschr. Naturforsch. Ges. Zürich 54, 309–329 (1909)]. However, he describes a kinematical construction for spherical rotors, and a method of successive approximation which would suffice for their practical use as shapes for special cams.

H. S. M. Coxeter (Toronto, Ont.).

Verblunsky, S. On the shortest path through a number of points. Proc. Amer. Math. Soc. 2, 904–913 (1951).

Let L_n denote the maximum of the length of the shortest path through n points of the unit square. It has been conjectured by the reviewer [Math. Z. 46, 83–85 (1940); these Rev. 1, 263] that

$$L_n=(4/3)^{1/4}n^{1/2}+o(n^{1/2})=(1.07\cdots)n^{1/2}+o(n^{1/2}).$$

In the present paper it is shown that

$$L_n<(2.8n)^{1/2}+2=(1.67\cdots)n^{1/2}+2.$$

L. Fejes Tóth (Veszprém).

Fejes Tóth, László. Über den Affinumfang. Math. Nachr. 6, 51–64 (1951).

In the first section of this paper, a variety of properties of the affine length are discussed. The following new definition is given: A "unit ellipse" is the image of the unit circle under a unimodular affine transformation. The affine length of an arc of a unit ellipse is the ordinary length of its original. Suppose the plane curve K has a continuous curvature everywhere. Decompose K into a finite number of subarcs k and replace each k by an arc e of a unit ellipse such that k and e have the same ordinary lengths and such that some points of k and e have the same curvature. If the maximum length of the k's converges to zero, then the sum of the affine lengths of the e's will converge to the affine length of K.

§2 deals with generalizations of the affine isoperimetric inequality $\lambda^3\leqq 8\pi^2 J$ for convex domains T with the area J and the affine perimeter λ. Theorem 2: If T is inscribed into a convex polygon with n sides, then $\lambda^3\leqq 8Jn^2\sin^2\pi/n$. In theorem 1, T is replaced by a convex arc AB inscribed into a given triangle OAB and such that the convex closure of AB has a given area. §3. Given a polygon U with the area $\mathfrak{U}$ and not more than six sides. Let Λ be the sum of the affine perimeters of n non-overlapping convex domains contained in U. Then $72\,\mathfrak{U}-\Lambda^3/n^2\geqq 0$ and $\liminf_{n\to\infty}(72\,\mathfrak{U}-\Lambda^3/n^2)=0$. In each of these theorems, the equality sign is discussed. The author indicates three-dimensional analogues of some of his results.

P. Scherk (Saskatoon, Sask.).

Green, John W. A note on the chords of a convex curve. Portugaliae Math. 10, 121–123 (1951).

Let C be a convex curve of width w in the euclidean plane; and let m be the maximum length of those chords of C subtended by orthogonal supporting lines of C. It is shown that $m\geqq w/\sqrt{2}$ with equality if and only if C is a circle.

W. Gustin (Bloomington, Ind.).

Macbeath, A. M. A compactness theorem for affine equivalence-classes of convex regions. Canadian J. Math. 3, 54–61 (1951).

Let C be the set of all convex bodies K in Euclidean n-space with inner points. Then C is a metric space and induces a topology in C^*, the set of equivalence classes of affinely related convex bodies. Theorem: C^* is a compact metric space. The metrization is accomplished by using

$$\Delta(K,K')=\log\rho(K,K')+\log\rho(K',K).$$

Here $\rho(K,K')$ is a continuous, affinely invariant function given by inf $V(\sigma K')/V(K)$, where $\sigma K'$ ranges through all convex bodies which are affinely equivalent to K' and contain K, and $V(K)$ is the volume of K. A supplementary result is that $\rho(K,K')\leqq n!n^n$.

L. Tornheim.

✓ ***Szegö, G. On certain set functions defined by extreme properties in the theory of functions and in mathematical physics.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 253–257. Amer. Math. Soc., Providence, R. I., 1952.

Rapport sur les progrès récents réalisés dans la solution de certains problèmes isopérimétriques. Pour un domaine plan, il s'agit, par example, de la comparaison entre l'aire, la longueur, les rayons conformes, intérieur et extérieur, la fréquence fondamentale d'une plaque encastrée, la rigidité d'une poutre admettant le domaine donné comme section droite, le diamètre transfini, etc.; dans l'espace, il s'agit de comparer, le volume, la surface, la capacité, etc. Les résultats sont dûs a l'auteur, à Pólya, à Schiffer.

J. Favard.

***Szegö, G. Principal frequency, torsional rigidity and electrostatic capacity.** Proc. Second Canadian Math. Congress, Vancouver, 1949. pp. 45–48. University of Toronto Press, Toronto, 1951. $6.00.

Les notions mentionnées dans ce titre, relatives à l'intérieur d'une courbe plane de Jordan, ont été, ainsi que bien d'autres dans ce même cas ou pour les domaines de l'espace, beaucoup étudiés et comparées depuis quelques dizaines d'années. L'auteur en donne un aperçu qui résume l'ouvrage d'ensemble qu'il vient de publier avec G. Pólya [Isometric inequalities in mathematical physics, Princeton Univ. Press, 1951; ces Rev. **13**, 270]. *M. Brelot* (Grenoble).

Algebraic Geometry

Gallarati, Dionigi. Sopra una notevole superficie del 6° ordine. Boll. Un. Mat. Ital. (3) **6**, 213–215 (1951).

L'auteur étudie une surface irréductible du sixième ordre qui a un point quadruple O: $A_4x_0^2+2B_5x_0+C_6=0$, A_4, B_5, C_6 étant des formes en x_1, x_2, x_3 de degré égal à l'indice. Il démontre qu'une telle surface possède au plus 45 points doubles isolés, et que ce nombre est atteint lorsque la courbe de diramation $B_5^2-A_4C_6=0$ est décomposée en dix droites qui sont dix bitangentes à la quartique de genre trois $A_4=0$, convenablement choisies. *L. Gauthier* (Nancy).

Baldassarri, Mario. Su un criterio di riduzione per un sistema algebrico di varietà. Rend. Sem. Mat. Univ. Padova **19**, 1–43 (1950).

U. Morin, généralisant un théorème de Noether et Enriques a donné une condition nécessaire et suffisante pour qu'un système à r paramètres de coniques tracées sur une V_{r+1} admette une variété unisécante [Rend. Sem. Mat. Univ. Padova **9**, 123–139 (1938)]. F. Conforto a étendu la condition suffisante à un système de quadriques F^2 [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 268–281 (1941); ces Rev. **8**, 223]. Dans le présent mémoire l'auteur donne une condition nécessaire et suffisante pour qu'un système algébrique à ρ paramètres de W_r^n, d'indice $\geqq 1$, qui engendre une $V_{\rho+r}$ admette une variété unisécante. Cette condition s'écrit:

$$A=r+2-n^\rho>0$$

sous réserve que $V_{\rho+r}$ soit "générale" dans un sens assez subtil, mais précisé. Lorsque $V_{\rho+r}$ n'est pas "générale", la condition $A>0$ est encore suffisante, mais n'est nullement nécessaire.

Voici comment on accède à cette notion de "variété générale": Soient K un domaine universel, et le corps $K'=K(u)$ mod φ, où $\varphi=0$ est l'équation d'une hypersurface Φ d'ordre μ de l'espace projectif $S_{\rho+1}/K$. On effectue sur cet espace la transformation de Veronese de degré t: $\Phi\rightarrow\Phi_t$ dans $S_{N(t)}$. La variété $V_{\rho+r}$ est représentée par le corps $K'(x)$ mod f, où $f=0$ est l'équation d'une hypersurface F' d'ordre n dans S'_{r+1}/K', dont les coefficients sont, dans $S_{\rho+1}/K$, d'ordre ν.

Un point rationnel de $V_{\rho+r}$ est obtenu en prenant les coordonnées x comme formes linéaires des coordonnées U dans un $S_{N(t)}$, t étant convenablement choisi:

$$x_i=a_i^jU_j$$

et on doit avoir $f=0$ mod φ dans $S_{\rho+1}/K$. Le quotient f/φ est alors d'ordre $nt+\nu-\mu$, et si S_M/K est l'espace des formes de ce degré dans $S_{\rho+1}/K$, l'identification conduit à un système:

$$\Psi_h(a_i^j)=L_h(b_h)\quad h=1, 2, \cdots, p,$$

où les Ψ_h sont des formes de degré n par rapport aux a_i^j et les L_h des formes linéaires par rapport aux coordonnées b_h de S_M/K. Ce système $\Psi_h-L_h=0$, indépendant de t pour t assez grand, est la base d'un idéal qui représente dans l'espace produit $(a, b)/K$ une variété W. En substituant aux Ψ_h et L_h des formes génériques ($/K$) de leurs degrés, on obtient un idéal premier qui représente une variété irréductible W^* de dimension d. La variété $V_{\rho+r}$ est "générale" lorsque W, éventuellement réductible, est pure, à composantes de dimension d. L'auteur montre par un exemple que cette notion de "généralité" n'est pas très restrictive; puis il donne diverses interprétations géométriques de son théorème. Il se propose de reprendre ultérieurement l'étude des variétés non "générales". *L. Gauthier* (Nancy).

***Segre, Beniamino. Arithmetical properties of algebraic varieties.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 490–493. Amer. Math. Soc., Providence, R. I., 1952.

Expository lecture. See the author's recent book "Arithmetical questions on algebraic varieties" [Athlone Press, London, 1951; these Rev. **13**, 273]. *D. Pedoe* (London).

***Zariski, Oscar. The fundamental ideas of abstract algebraic geometry.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, 77–89. Amer. Math. Soc., Providence, R. I., 1952.

This very clear and comprehensive report deals with the impact of the ideas and methods of modern algebra on algebraic geometry during the past 25 years, and with the substantial advance thus derived for algebra, to which Zariski himself has largely contributed. The main topics are: the concept of an algebraic variety, the theory of specializations, the concept of a normal variety, holomorphic functions and the principle of degeneration, valuation theory. It is rather a pity that there is no bibliography.

The algebraic varieties considered here are supposed to be immersed in projective spaces over a universal domain, given once for all, i.e., an algebraically closed field having infinite transcendence degree over some of its subfields k, the allowable fields. If a variety V admits a system of defining equations with coefficients in k, then k is said to be a field of definition of V; and a property of V is called relative or absolute according as it does or does not depend on the choice of the field of definition of V. Then some of the set-theoretic and topological implications of these notions are discussed; and a distinction is indicated between general and generic points of an irreducible variety. Next it is said that the elementary theory of algebraic varieties, including the theory of specializations and their extensions, can be founded on Hilbert's Nullstellensatz, and that a general intersection theory can be derived from the theory of specializations. Many open questions on the subject are also pointed out.

The important notion of normal varieties (in the sense of Zariski) is then recalled, and some of its applications are given. Roughly speaking, in the case of normal varieties the presence of an isolated specialization implies the uniqueness of that specialization; and this property leads to the proof of the uniqueness of the intersection multiplicity of two varieties at a common isolated intersection. Any variety V can be transformed into a normal variety by some birational transformation having no fundamental points on V. If V is normal and has dimension r, its singular locus is of dimension at most $r-2$; moreover, V has then the characteristic property that the hypersurfaces of its ambient space, of a

sufficiently high order, cut out on V a complete linear system. Defining the virtual arithmetic genus $p(V)$ of V as the constant term in Hilbert's postulation formula of V, supposing $r\leqq 3$, and assuming V variable in a birational class, it can be shown that $p(V)$ has a minimum, reached for the non-singular varieties of the class and given by the effective arithmetic genus of V.

The abstract notion of connectedness of an algebraic cycle over a field k is afterwards given, and the ring o_W^* of functions over k on an algebraic variety V—which are holomorphic along a subvariety W of V—is introduced. If V is analytically irreducible at each point of W, then W is connected if and only if o_W^* is an integral domain. By combining this criterion with a theorem of invariance of the ring o_W^* with respect to algebraic transformations of V, there follows a remarkable connectedness theorem on algebraic correspondences; the latter gives Enriques' principle of degeneration as an immediate and particular consequence.

In conclusion the valuation theory is considered as a transcendental theory of specializations, apt to deal with the finer differential aspects of the local geometry on a variety, especially in relation to the problems of local uniformization and of the resolution of singularities.

B. Segre (Rome).

Zariski, Oscar. Sur la normalité analytique des variétés normales. Ann. Inst. Fourier Grenoble **2** (1950), 161–164 (1951).

Let V be an algebraic variety and P a point of V at which V is locally normal; denote by $\mathfrak{o}$ the ring of the point P, which is therefore integrally closed. Then it is proved that the completion $\bar{\mathfrak{o}}$ of $\mathfrak{o}$ (which is already known to be a domain of integrity) is still integrally closed; in other words, P is still normal on V if we consider V as an algebroid variety. In order to establish this, the author proves the following theorem. Let $\mathfrak{o}$ be a local domain of integrity which is integrally closed, and let $\bar{\mathfrak{o}}$ be the completion of $\mathfrak{o}$. Denote by A the ring of quotients of $\bar{\mathfrak{o}}$ and by $\bar{\mathfrak{o}}'$ the integral closure of $\bar{\mathfrak{o}}$ in A. Assume that there exists a $d\neq 0$ in $\mathfrak{o}$ with the following properties: $d\bar{\mathfrak{o}}'\subset\bar{\mathfrak{o}}$, and the prime divisors of $d\mathfrak{o}$ are analytically unramified. Then we have $\bar{\mathfrak{o}}=\bar{\mathfrak{o}}'$ and $\bar{\mathfrak{o}}$ is a domain of integrity. The proof makes use of a certain number of preliminary lemmas of the proof of the local irreducibility of a normal variety, but does not make use of this theorem itself. As a matter of fact, it is shown that the theorem in question may be deduced easily from the theorem stated above, which simplifies its proof. Finally, the author establishes that, for the local ring of a point P at which the variety V is not necessarily locally normal, the operations of taking the integral closure and of taking the completion commute with each other.

C. Chevalley.

Samuel, Pierre. Sur les variétés algébroïdes. Ann. Inst. Fourier Grenoble **2** (1950), 147–160 (1951).

The author proves that an algebroid variety V of dimension $\geqq 2$ whose hyperplane sections are all algebraic is itself an algebraic variety. It is clearly sufficient to establish this in the case where V is a hypersurface. In that case, the author first proves that, if there exists a pencil of hyperplanes whose base is not on V and whose members intersect V along algebraic cycles, then V is a sheet of an algebraic variety. The theorem then follows from this lemma by applying it to several properly selected pencils. It is also proved that, in general, the set of hyperplanes whose intersections with V are algebraic is representable as the union of a countable family of algebraic varieties of hyperplanes (this family may actually be infinite). Then the author shows how his results allow one to simplify some parts of the proof of Chow's theorem which states that a compact analytic manifold in projective space is an algebraic variety.

The third part of the paper is concerned with the proof of the following theorem. Let A be an algebraic variety, N a subvariety of A and X a divisor on A such that those components of X which contain N are simple on A. Assume that, on a neighbourhood of N, X may be represented as the complete intersection of A with some algebroid divisor; then it can also be represented (along N) as the complete intersection of A with an algebraic divisor. In an appendix, it is established that, $\mathfrak{a}$ and $\mathfrak{b}$ being ideals in a local ring $\mathfrak{o}$, and $\bar{\mathfrak{o}}$ the completion of $\mathfrak{o}$, then $\mathfrak{a}\bar{\mathfrak{o}}\cap\mathfrak{b}\bar{\mathfrak{o}}=(\mathfrak{a}\cap\mathfrak{b})\bar{\mathfrak{o}}$.

In the proof of lemma 1 in part 3, no reference is given to establish that there actually exists an element x with the properties postulated in the proof.

C. Chevalley.

∗**de Rham, Georges. Intégrales harmoniques et théorie des intersections.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 209–215. Amer. Math. Soc., Providence, R. I., 1952.

An expository account of the application of the theory of harmonic forms to the problem of the intersection of chains on a manifold, as expounded in detail in "Harmonic Integrals" by de Rham and Kodaira [Institute for Advanced Study, Princeton, N. J., 1950; these Rev. **12**, 279]. The relevant part of the theory of currents is explained, together with the decomposition theorem for currents. This is then applied to obtain the formula

$$I(c^p, c^{n-p})=\int_{c_y^{n-p}}\int_{c_x^p}e(x, y^*)-\int_{c_x^p}\int_{c_y^{n-p}}e(x, y^*)$$

for the Kronecker index of two chains c^p, c^{n-p}, neither of which meets the boundary of the other, where $e(x, y)$ is a well-defined double p-form which is C^∞, except at $x=y$ where it has a singularity of order n.

W. V. D. Hodge.

∗**Weil, André. Number-theory and algebraic geometry.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 90–100. Amer. Math. Soc., Providence, R. I., 1952.

Ici nous trouvons à la fois un manifeste et un programme. Comme dans plusieurs de ses écrits récents, introductions ou analyses, l'auteur s'insurge contre l'emprise absolue de l'algèbre "pure" sur ses applications; et, s'opposant à la nombreuse lignée de Dedekind et d'Emmy Noether, il met l'accent sur les idées, un peu oubliées, de Kronecker. Ainsi sont passés en revue les quelques résultats modernes qui font partie du programme Kroneckérien, et les nombreux problèmes que pose la réalisation de celui ci.

Est d'abord examinée la notion de dimension, définie comme longueur d'une suite de spécialisations succéssives. On voit que, dans toute question de géométrie algébrique, on peut, par passage à une variété de dimension supérieure, opérer sur une extension algébrique du corps premier. D'autre part les spécialisations d'inégales caractéristiques accroissent d'une unité la dimension des corps de caractéristique nulle; par exemple les corps de dimension 1 sont les corps de nombres algébriques et les corps de fonctions algébriques d'une variable sur un corps fini, cas bien étudiés, dont l'analogie est féconde, mais qui présentent de nombreux problèmes non résolus, comme ceux relatifs à la fonction zèta. Plus difficile encore semble la généralisation de la fonction zèta aux cas de dimension supérieure; une

définition naturelle en est donnée, et plusieurs conjectures faites à son sujet.

Mais les questions locales introduisent d'autres corps et anneaux de base que les corps absolument algébriques; ce sont les anneaux locaux complets en général, les p-adiques en particulier, et pour étudier les places à l'infini, les corps réel et complexe (qu'un arithméticien aurait ainsi été amené à étudier, si l'histoire n'avait suivi la marche inverse). Divers résultats appartenant à la géométrie sur les anneaux et corps locaux sont passés en revue, ceux de Skolem, Chabauty et Lutz dans le cas p-adique, ceux de Zariski et Chow dans le cas général. On est ici un peu surpris de l'hommage rendu à ceux des fidèles de Dedekind et E. Noether qui ont développé la théorie des anneaux locaux dans le cadre de l'algèbre pure.

Le programme Kronéckérien vise, entre autres choses, à la création d'une géométrie algébrique sur les entiers. Un exemple typique est l'étude, par Kronecker lui même, d'une correspondance sur une surface dont on déduit de nombreux résultats sur les transformations des fonctions elliptiques, la multiplication complexe par exemple.

L'auteur montre enfin comment la définition Kronéckérienne des idéaux par adjonction d'indéterminées est réliée aux "formes associées" de Chow, à la théorie des séries linéaires, et à la notion de degré d'une variété. L'analogue arithmétique de cette notion de degré est celle de "hauteur" d'un point algébrique [Ann. of Math. (2) **53**, 412–444 (1951); ces Rev. **13**, 66] essentielle dans les travaux de l'auteur, de Siegel, de Northcott et d'autres. Il est rappelé comment la théorie des "distributions" (ibid.) permet d'étudier cette notion, et d'établir une inégalité fondamentale entre les hauteurs de deux points algébriques se correspondant par transformation birationnelle.

P. Samuel (Clermont-Ferrand).

Differential Geometry

Kubota, Tadahiko. Einige Sätze über Kinematik. Proc. Japan Acad. **24**, no. 7–8, 4–6 (1948).

The author gives an analytical proof of the cinematical theorem of Mannheim concerning 4 points on a moving line, which are restrained to move on 4 spheres with coplanar centers. Two generalizations, involving 5 and 6 points, are proved. *H. Samelson* (Ann Arbor, Mich.).

Horninger, H. Über eine planare Evolventenbewegung (zylindrische Rollung einer Ebene). Bull. Tech. Univ. Istanbul **3** (1950), no. 1, 103–122 (1951). (German. Turkish summary)

Die schiefen Regelschraubflächen ϕ können bekanntlich nicht nur durch Schraubung, sondern auch durch zylindrische Rollung erzeugt werden. Die feste Polfläche ist dabei ein Drehzylinder T, die bewegte Polfläche eine Tangentialebene t von T; jede zu t parallele Gerade durchläuft die Erzeugendenschar einer Fläche ϕ. Alle Punkte des durch t repräsentierten bewegten System Σ bewegen sich auf Kreisevolventen, deren Ebenen zueinander parallel sind. Verf. untersucht nun die Bahnflächen beliebiger Geraden und die Hüllflächen beliebiger Ebenen von Σ.

O. Bottema (Delft).

Roşca, Radu. On transformations of quadratic curves. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. **2**, 647–652 (1950). (Romanian. Russian and French summaries)

Tzitzéica [Introduction à la géométrie différentielle projective des courbes, Mémor. Sci. Math., no. 47, Gauthier-Villars, Paris, 1931] studies two types of transformations of quadratic curves into quadratic curves, using (a) a fixed point; (b) punctual antiderivatives of the first order. In a previous unavailable memoir the author studied asymptotic transformations [see Bianchi, Rend. Circ. Mat. Palermo **25**, 291–325 (1908)] and their compositions with the first type of Tzitzéica's transformations. Main object of the present paper is the study of the corresponding compositions with the second type of Tzitzéica's transformations. Several results are established of which the simplest reads as follows: In the projective euclidean three-space consider the two quadratic curves (x), $(\bar{x})$ given in asymptotic correspondence. If they are transformed by their first order antiderivatives (a), $(\bar{a})$, which are also taken to be in asymptotic correspondence and satisfying, furthermore, the relation $cSa^2+\bar{c}S\bar{a}^2-2Sa\bar{a}=0$ (c, c' constants), then the transformed quadratic curves (y), $(\bar{y})$ are also in asymptotic correspondence. *E. Grosswald* (Princeton, N. J.).

Blaschke, W. Geometria affine. Matematiche, Catania **5**, 32–44 (1950); **6**, 42–50 (1951).

Expository lectures held at the University of Catania, April 1950.

Norden, A. P. On a geometric characteristic of a mapping with the aid of analytic functions of two complex arguments. Doklady Akad. Nauk SSSR (N.S.) **81**, 145–147 (1951). (Russian)

The author considers a four-dimensional space whose fundamental group is the subgroup of affine transformations leaving invariant two infinitely distant conjugate complex lines. Such a space is a biaffine B_4. The directions at a point of B_4 form a projective space P_3 in which is determined a biaxial angular metric which is determined by an absolute involution defined by a tensor G_j^i subject to $G_j^iG_k^j=-\delta_k^i$, $G_k^k=0$. Conjugate vectors are then determined by $\bar{a}^i=G_k^ia^k$. The author considers motions in B_4 which form a 12-parameter group and by mapping B_4 on a plane A_2 with complex coordinates the isomorphism of this group with the affine group of this plane is established. A biconformal mapping of B_4 on itself is defined as a mapping preserving G_j^i; when this is established in canonical coordinates, such a mapping is given by

$$y_1+iy_2=\varphi(x_1+ix_2, x_3+ix_4) \quad \text{and} \quad y_3+iy_4=\psi(x_1+ix_2, x_3+ix_4),$$

where φ and ψ are analytic functions of two complex variables. *M. S. Knebelman* (Pullman, Wash.).

Debever, Robert. Les espaces de l'électromagnétisme. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 217–233. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs; 2450 French francs.

This paper deals with a 4-space without metric in which an electromagnetic field is defined by a covariant skew-symmetric tensor H_{rs} and a contravariant skew-symmetric tensor density $\mathfrak{K}^{rs}$, or equivalently by two covariant tensors H_{rs}, K_{rs} where $K_{rs}=\frac{1}{2}\epsilon_{rsmn}\mathfrak{K}^{mn}$. The author notes connections between the present work and a paper by the reviewer [Proc. Symposia Appl. Math., vol. 2, pp. 21–48, Amer. Math. Soc. New York, 1950; these Rev. **11**, 401] dealing

with same situation but in a different notation; in the present paper Cartan's exterior forms are used. Among other results, the author claims to answer in the affirmative the following question raised by the reviewer in the paper cited. Given that the following invariant relations hold in a domain R of the 4-space: $H\times H+K\times K=0$ (where $H\times H=\frac{1}{4}e^{rstu}H_{rs}H_{tu}$), $H_{[mn,r]}=0$, $K_{[mn,r]}=0$, does there exist a coordinate system in R such that

$$\begin{aligned} &H_{23}=K_{14}, &&H_{31}=K_{24}, &&H_{12}=K_{34},\\ &H_{14}=-K_{23}, &&H_{24}=-K_{31}, &&H_{34}=-K_{12}? \end{aligned}$$

The proof is given in very brief form, but it seems to the reviewer that it requires the use of complex coordinates in R, thus raising issues that need clarification before the answer can be regarded as convincingly established. There are a number of misprints in the paper. *J. L. Synge.*

Petrov, A. Z. On spaces defined by a gravitational field. Doklady Akad. Nauk SSSR (N.S.) **81**, 149–152 (1951). (Russian)

An Einstein space V_n with fundamental tensor g_{ij} is characterized by $R_{ij}=xg_{ij}$, where R is the Ricci tensor. In order that such a space for $n=4$ should have a real gravitational field it is necessary and sufficient that the local geometry at any point be Minkowskian. Many special solutions of this problem are known and the present paper gives a complete classification of such spaces, denoted by T_4. This is accomplished by assigning to an antisymmetric pair of indices $[ij]$ a single index α, arranged in some lexicographic order so that to each point P of T_4 corresponds a local centro-affine geometry of Klein in a phase space of $\frac{1}{2}n(n-1)=6$ dimensions with the group $\eta^{\alpha'}=A_\alpha{}^{\alpha'}\eta^\alpha$, $|A_\alpha{}^{\alpha'}|\neq 0$ where

$$A_\alpha{}^{\alpha'}\rightarrow 2A_{[ij]}^{i'j'},\quad A_i{}^{i'}=\left(\frac{\partial x^{i'}}{\partial x^i}\right)_P.$$

The metric for bivectors is given by $g_{ik}g_{jl}-g_{il}g_{jk}\rightarrow g_{\alpha\beta}$ and $|g_{\alpha\beta}|\neq 0$ since it is the square of the determinant $|g_{ij}|$. Similarly, the curvature for any orientation may be written as $K=R_{\alpha\beta}V^\alpha V^\beta/g_{\alpha\beta}V^\alpha V^\beta$ where $R_{\alpha\beta}\rightarrow R_{ijkl}$. The stationary values of K are given by $(R_{\alpha\beta}-Kg_{\alpha\beta})V^\beta=0$ and the classification of T_4 is thus reduced to the classification of a pair of tensors in 6-dimensional space in which $g_{\alpha\beta}$ is nonsingular and indefinite. An examination of the characteristic roots shows that the matrix $\|R_{\alpha\beta}-Kg_{\alpha\beta}\|$ belongs to one of the two categories: [111 111] or [(3 3)] and the first of these may be of one of the following classes: for real roots 1) [(11) (11) (11)], 2) [(1111) (11)], 3) [(111 111)]; for complex roots; 4) $[\overline{11}\ \overline{11}\ \overline{11}]$, 5) $[\overline{(11)\ (11)}\ \overline{11}]$, 6) $[\overline{(11)\ (11)}\ (11)]$ and for the second category 7) [(3 3)]. This classification leads to the canonical form of $g_{\alpha\beta}$ for some of the classes, but such a form for g_{ij} is given only for the second category.

M. S. Knebelman (Pullman, Wash.).

Tachibana, Syun-ichi. On concircular geometry and Riemann spaces with constant scalar curvatures. Tôhoku Math. J. (2) **3**, 149–158 (1951).

The author reviews basic concircular geometry [cf. K. Yano, Proc. Imp. Acad. Tokyo **16**, 195–200, 354–360, 442–448, 505–511 (1940); **18**, 446–451 (1942); **19**, 444–453 (1943); these Rev. **2**, 165, 303; **7**, 330] and some of the theory of conformal connections, and then introduces spaces Ω_n with conformal connections whose conformal curvature tensors satisfy

$$F^i{}_{jkl}=Z^i{}_{jkl}\equiv R^i{}_{jkl}-\frac{R}{n(n-1)}(g_{jk}\delta_l{}^i-g_{jl}\delta_k{}^i),$$

$Z^i{}_{jkl}$ being the "concircular curvature tensor". These spaces are in one-to-one correspondence with classes of Riemann spaces, each space of one class concircular to the others of that class, and if the holonomy group of a Ω_n fixes a point (or a hypersphere) then the Ω_n corresponds to a concircular class of Riemann spaces, including one with non-vanishing (or vanishing) constant scalar curvature, and conversely. This generalizes the corresponding theorems of S. Sasaki on spaces with normal conformal connections [cf. Jap. J. Math. **18**, 615–622, 623–633, 791–795 (1943); these Rev. **7**, 330]. Sasaki also proved that the Poincaré representation for non-Euclidean geometries could be generalized to Einstein spaces with non-vanishing scalar curvature. This is now further generalized to Riemann spaces with non-vanishing constant scalar curvature.

The author states theorems on concircular transformations of Riemann spaces with constant scalar curvature. For instance, if a Riemann space with constant scalar curvature c is non-trivially concircular to a second Riemann space with constant scalar curvature $\bar{c}$, then it is also non-trivially concircular to a Riemann space with any preassigned constant scalar curvature $\bar{\bar{c}}$. The paper concludes with a canonical form for the line element of a Riemann space of constant scalar curvature which is non-trivially concircular to another such space. The proofs of most of the theorems are not given since they follow rather closely the corresponding proofs in the work of Sasaki and Yano and Mutô [cf. papers cited above and also K. Yano and Y. Mutô, J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **4**, 117–169 (1941); these Rev. **3**, 192]. *A. Schwartz* (New York, N. Y.).

Bompiani, E. Géométries riemanniennes d'espèce supérieure. Colloque de Géométrie Différentielle, Louvain, 1951, pp. 123–156. Georges Thone, Liège; Masson & Cie, Paris, 1951. 350 Belgian francs; 2450 French francs.

The first part of this paper is a summary of the author's theory of Riemannian spaces imbedded in Euclidean spaces, the chief feature of which is his use of osculating spaces of various orders. Comparison is made with the similar theories developed by Vitali, Bortolotti, and Burstin-Mayer. Using the author's theory it is possible to define new types of displacements for vectors which (for example) will take a vector of a given osculating space and keep it in that space. Such displacements no longer preserve length and angle. The remainder of the paper is concerned with deformations of spaces which preserve not only the intrinsic metric but also the higher curvatures up to a stated order. There is a detailed discussion of the projective properties of such deformations for two-dimensional surfaces.

C. B. Allendoerfer (Seattle, Wash.).

Kimpara, Makoto. Sur les réseaux plans dans un espace à connexion projective à deux dimensions. Tôhoku Math. J. (2) **3**, 174–181 (1951).

Let P_2 be a two-dimensional projective space referred to the coordinates x^i ($i, j, k=1, 2$) and let

$$x^i=x^i(u^1, u^2) \tag{1}$$

be the parametric equations of a net. Put

$$\underset{j}{e}^i=\partial x^i/\partial u^j\quad \underset{j}{e}^0=0\quad \underset{0}{e}^\nu=\delta_0{}^\nu\quad (\nu, \lambda, \mu=0, 1, 2).$$

Then the connection of P_2 is involved in

$$D_j\underset{0}{e}^\nu=\underset{j}{e}^\nu\quad D_j\underset{k}{e}^\nu=\Omega^\lambda{}_{jk}\underset{\lambda}{e}^\nu \tag{2}$$

which yields at once the invariants

$$\Omega^2{}_{11}\Omega^1{}_{22}du^1du^2,$$

$$\tfrac{1}{2}\Omega^2{}_{11}\Omega^1{}_{22}(\Omega^2{}_{11}(du^1)^3+\Omega^1{}_{22}(du^2)^3),$$

$$\frac{\Omega^2{}_{11}(du^1)^3+\Omega^1{}_{22}(du^2)^3}{du^1du^2},$$

the last one being the linear projective element ds. It may be easily characterized by means of a cross ratio and the usual cubic curve. The invariants

$$h=\Omega^0{}_{21}+\Omega^1{}_{21}\Omega^2{}_{21}-\frac{\partial\Omega^1{}_{21}}{\partial u^1},$$

$$k=\Omega^0{}_{12}+\Omega^2{}_{12}\Omega^1{}_{12}-\frac{\partial\Omega^2{}_{12}}{\partial u^2}$$

may also be characterized geometrically by means of the Laplace transforms. In particular, $h=k$ if and only if a certain projectivity between $\underset{0}{e}$ and the line which joins the Laplace transforms of $\underset{0}{e}$ is a polarity. The author finds also the geometrical interpretations of the vanishing of the curvature tensor of Ω by means of the net (1). This brief account does not exhaust all results proved by the author.

V. Hlavatý (Bloomington, Ind.).

Yano, Kentaro, and Hiramatu, Hitosi. Affine and projective geometries of system of hypersurfaces. J. Math. Soc. Japan 3, 116–136 (1951).

Dans un espace n-dimensionel X_n, rapporté aux coordonnées (x^i), les auteurs supposent donné un système d'hypersurfaces $f(x, a)=0$ où f dépend de n paramètres essentiels a^i. La géométrie de ce système d'hypersurfaces est étudiée relativement aux transformations de coordonnées et à la transformation $\bar{f}=\alpha f$. Si α ne dépend que des a, la géométrie est dite affine; elle est dite projective pour $\alpha(n, a)$. À la donnée de f on peut substituer celle d'un système de Pfaff complètement intégrable

$$u_i dx^i, \quad du_j-\Gamma_{jk}(x, u)dx^k=0,$$

où (x^i, u_i) définissent les éléments plans tangents aux hypersurfaces. La géométrie d'une tel système a été étudiée antérieurement dans le cas de variétés de dimensions quelconques par Chern [Proc. Nat. Acad. Sci. U. S. A. 29, 38–43 (1943); ces Rev. 4, 259] et Yen Chih-Ta [C. R. Acad. Sci. Paris 227, 461–462 (1948); ces Rev. 10, 211] par des méthodes différentes. Ici les auteurs construisent, dans le cas affine, une connexion affine et un tenseur de courbure; ils établissent une condition nécessaire et suffisante pour que f soit réductible à une forme linéaire en x^i. Le même problème est résolu dans le cas projectif, grâce à cette étude préalable. Après un rappel de la notion de connexion projective (au sens de Cartan) sur un espace d'éléments plans (x^i, u_i), les auteurs arrivent au problème principal de ce travail: X_n étant envisagé comme espace d'éléments plans, douer cet espace d'une connexion projective telle que le système des hypersurfaces projectivement planes associé à la connexion coincide avec le système des hypersurfaces $f=0$ donné. Ce problème est entièrement résolu et les composants de la connexion cherchée dans un repère naturel (au sens de Cartan) sont données en fonction des Γ_{jk}.

A. Lichnerowicz (Paris).

Kawaguchi, Michiaki, Jr. On the theory of a rheonomic Cartan space. J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 167–179 (1950).

Rheonomic Cartan space is a generalization of Cartan space as treated in Berwald [Acta Math. 71, 191–248 (1939); these Rev. 1, 177] in that the $(n-1)$-dimensional area is assumed to depend on the time. If the hypersurface is given by $x^\alpha=x^\alpha(v^1, \cdots, v^{n-1}, t)$, $\alpha=1, \cdots, n$, its measure of $(n-1)$-dimensional time-area is given by

$$\int\psi\left(x^\alpha, \frac{\partial x^\alpha}{\partial x^i}, \frac{\partial x^\alpha}{\partial t}, t\right)dv^1\cdots dv^{n-1}dt, \quad i=1, \cdots, n-1.$$

Since this integral must be invariant for transformations of parameters of the form $\bar{v}^i=\bar{v}^i(v^1, \cdots, v^{n-1}, t)$ the integrand can be written in the form $L(x^\alpha, t, u_\alpha, u_0)$ where u_α has the meaning given in Berwald, and where $u_0=-(\partial x^\alpha/\partial t)u_\alpha$. With L as fundamental function, metric tensor, connection coefficient and curvature tensors are introduced. When t is constant the spaces reduce to ordinary Cartan spaces.

E. T. Davies (Southampton).

Aussem, M. V. Metric spaces of n dimensions based on the concept of the area of an m-dimensional surface. Doklady Akad. Nauk SSSR (N.S.) 80, 701–704 (1951). (Russian)

The author generalizes formally the connection theory of E. Cartan [Les espaces métriques fondés sur la notion d'aire, Actualités Sci. Ind., no. 72, Hermann, Paris, 1933] to areal spaces of the metric class. Let an m-dimensional surface be given by $x^\alpha=f^\alpha(x^i)$ and its area by

$$\sigma=\int_{(m)}F(x^I, p_i^\alpha)[dx^1, \cdots, dx^m]$$
$$=(\prod_\gamma u_\gamma{}^\gamma)^{-1}\int_{(m)}L(x, u)[dx^1\cdots dx^m],$$

putting $p_i^\alpha=\partial x^\alpha/\partial x^i=-u_i{}^\alpha/u_\alpha{}^\alpha$, where Latin indices run over $1, 2, \cdots, m$, Greek ones $m+1, \cdots, n$ and capital ones $1, 2, \cdots, n$. In the same way as in Cartan, the connection parameters Γ^K_{IH}, $C^{K(H)}_I$ can be obtained from invariance of the metric tensor g^{IJ} under mapping defining the connection and from symmetry of $C^{IJ(H)}$ and

$$\Gamma^K_{IJ}+C^{K(L)}_I p_{L_1\cdots L_{k-1}ML_{k+1}\cdots L_{n-m}}{}^M_{L_k J}$$

in I and J, where (H) denotes a system of indices $H_1, H_2, \cdots, H_{n-m}$. An areal element here is represented by an $(n-m)$-vector $p_{(H)}$ whose components are expressed by $u_i{}^\alpha$, $u_\alpha{}^\alpha$ (not summing) and L, and its transversal element at the same point by $L^{(H)}$ whose components are constructed of

$$L_{u_i{}^\alpha}, \quad L_{u_\alpha{}^\alpha}=-\frac{u_i{}^\alpha}{u_\alpha{}^\alpha}L_{u_i{}^\alpha}+\frac{L}{u_\alpha{}^\alpha}$$

and L such that $L^{(H)}p_{(H)}=L$, $L^{(H)}dp_{(H)}=dL$. Under the assumption that $L^{(H)}$ be perpendicular to $p_{(H)}$, i.e. $LL^{(I)}=gg^{(I),(J)}p_{(J)}$, it is derived that

$$gg^{(I),(J)}=L^{(I)}L^{(J)}+LL^{(I),(J)},$$

where $L^{(I),(J)}$ is defined by $dL^{(I)}=L^{(I),(J)}dp_{(J)}$, but has $\binom{n}{m}\{\binom{n}{m}-m(n-m)-1\}$ arbitrary parameters. In order that g^{IJ} be determined from $g^{(I),(J)}$, $gg^{(I),(J)}$ must satisfy a number of relations of which certain ones give us the differential equations of the function L, but their explicit forms are not stated. [Reviewer's note. The author seems not to be acquainted with the recent papers of R. Debever [Sur une classe d'espaces à connexion euclidienne, Thèse, Université Libre de Bruxelles, 1947; these Rev. 9, 379] and of the re-

viewer [Tensor N.S. 1, 14–45 (1950), 67–88, 89–103 (1951); these Rev. 12, 536; 13, 384, 385]. It must therefore be remarked that the partial derivatives $\partial g^{IJ}/\partial p_{(H)} = -2C^{IJ}_{(H)}$ have no meaning, because $p_{(H)}$ is a simple $(n-m)$-vector, when $n-1>m>1$.] *A. Kawaguchi* (Sapporo).

Moór, Arthur. Einführung des invarianten Differentials und Integrals in allgemeinen metrischen Räumen. Acta Math. 86, 71–83 (1951).

In a space $\mathfrak{R}_n$ of line elements $(x, \dot{x})$ the invariant differential of a contravariant vector field $\xi^i(x, \dot{x})$ along a curve $x^i = x^i(t)$, $\dot{x}^i = \dot{x}^i(t)$ is defined by

$$D\xi^i = {}_{[a]}\lambda^i d_{[a]}\Phi = d\xi^i + (\Gamma^i_{jk}dx^k + C^i_{jk}d\dot{x}^k)\xi^j$$

using the a priori given covariant n-ple ${}_{[a]}\mu_i$ and its conjugate ${}_{[a]}\lambda^i$, where ${}_{[a]}\Phi = {}_{[a]}\mu_k\xi^k$; and similarly for a covariant vector field $\xi_i(x, \dot{x})$. Then C^i_{jk} and Γ^i_{jk} satisfy the well-known relations

$$\partial g_{ij}/\partial x^h = g_{ik}C^k_{jh} + g_{jk}C^k_{ih} \quad \text{and} \quad \partial g_{ij}/\partial \dot{x}^h = g_{ik}\Gamma^k_{jh} + g_{jk}\Gamma^k_{ih},$$

provided that $g_{ij} = {}_{[a]}\mu_{i[a]}\mu_j$. In an analogous way, the author defines the invariant integral:

$$\zeta^i = {}_{[a]}\lambda^i \int_{t_0}^{t} {}_{[a]}\Phi dt \qquad \left(= \mathop{\mathrm{I}}\limits_{t_0}^{t} \xi^i dt \right)$$

and proves that the invariant integral is the inverse operation of the invariant differential. It must be noticed that the invariant integral is a curve integral along the curve $(x^i(t), \dot{x}^i(t))$ and that the n-ples ${}_{[a]}\mu_i$ and ${}_{[a]}\lambda^i$ behave like constants; for example,

$$D_{[a]}\mu_i/dt = 0 \quad \text{and} \quad \mathop{\mathrm{I}}\limits_{t_0}^{t} {}_{[a]}\lambda^i dt = {}_{[a]}\lambda^i(t-t_0).$$

There is an application of the invariant integral to the Frenet formulas of a curve. In the last paragraph the author generalizes these operations to a Kawaguchi space [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. 9, 1–152 (1940); these Rev. 2, 22; Proc. Imp. Acad. Tokyo 13, 183–186, 237–240 (1937)] by considering an n-ple of excovariant exvectors of order G: ${}_{[a]}\mu_{\beta j}$ and with the assumption of the existence of its conjugate ${}_{[a]}\lambda^{\gamma i}$ which satisfies the relation ${}_{[a]}\lambda^{\gamma i}{}_{[a]}\mu_{\beta j} = \delta_\beta{}^\gamma \delta_j{}^i$. Namely, the invariant differential and integral are given by

$$DV^{\alpha i} = {}_{[a]}\lambda^{\alpha i}d_{[a]}\Phi \quad \text{and} \quad \mathop{\mathrm{I}}\limits_{t_0}^{t} V^{\alpha i}dt = {}_{[a]}\lambda^{\alpha i}\int_{t_0}^{t} {}_{[a]}\Phi dt,$$

respectively, where $V^{\alpha i}$ is an exvector and ${}_{[a]}\Phi = {}_{[a]}\mu_{\beta j}V^{\beta j}$. Reviewer's remark: The connection defined by the above-stated invariant differential is that of fern-parallelism, that is, the curvature tensors are equal to zero. Consequently, the connection is essentially different from those of E. Cartan [Les espaces de Finsler, Actualités Sci. Ind., no. 79, Hermann, Paris, 1934] and of O. Varga [Monatsh. Math. Phys. 50, 165–175 (1941); these Rev. 5, 218]. In a Kawaguchi space it is better to take an $n(G+1)$-ple of exvectors ${}_{[\alpha a]}\mu_{\beta j}$ in place of an n-ple ${}_{[a]}\mu_{\beta j}$; then its conjugate ${}_{[\alpha a]}\lambda^{\gamma i}$ is determined uniquely from the relation ${}_{[\alpha a]}\lambda^{\gamma i}{}_{[\alpha a]}\mu_{\beta j} = \delta_\beta{}^\gamma \delta_j{}^i$ and the assumption of its existence is unnecessary.
A. Kawaguchi (Sapporo).

✓ ***Chern, Shiing-shen. Differential geometry of fiber bundles.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 397–411. Amer. Math. Soc., Providence, R. I., 1952.

The starting point of this lecture is the definition of a connection in a principal fiber bundle (all spaces are differentiable manifolds, the structure group is a Lie group G with Lie algebra $L(G)$), generalizing the well-known Levi-Civita parallelism. Geometrically the connection is a field of contact elements in the bundle, transversal to the fibers, and invariant under the action of the group. Covariant differentiation with respect to the connection is defined, a generalized curvature tensor, an exterior 2-form with values in $L(G)$, is constructed. With this tensor one can define a map h of the ring $I(G)$ of invariant symmetric multilinear forms on $L(G)$ into the cohomology ring $H(X)$ of the base space X, by substituting the curvature tensor into the arguments of the forms. The map h turns out to be essentially the same as the characteristic homomorphism obtained from the theory of universal bundles for G; this is the main result of the theory. An important step consists in showing that $I(G)$ and the cohomology ring of the classifying space (base space of the universal bundle) are isomorphic; this belongs to the theory of homogeneous spaces; actually one has to consider the maximal compact subgroup G_1 of G. The relation to the theory of transgression is studied. Several examples are discussed: (1) $G = R(m)$ = orthogonal group in m variables. The generators of $I(G)$ determine the Pontryagin classes and the Allendoerfer-Weil-Chern form; the classifying space is the Grassmann manifold, and one can explicitly set up the isomorphism between its cohomology ring and $I(R(m))$. (2) $G = U(m)$ = unitary group. This leads to the Chern classes of a hermitian manifold. (3) $G = GL(m)$ = general linear group. By studying $I(G)$ one finds that a Gauss-Bonnet formula is impossible. Some remarks are made concerning the relation to the characteristic cocycles (Stiefel-Whitney), the topological invariance of the characteristic homomorphism of a tangent bundle (Thom, Wu), and the concept of holonomy group.
H. Samelson (Ann Arbor, Mich.).

✓ ***Allendoerfer, Carl B. Cohomology on real differentiable manifolds.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 428–435. Amer. Math. Soc., Providence, R. I., 1952.

The theorems of de Rham [J. Math. Pures Appl. (9) 10, 115–200 (1931)] establish the relationship between the cohomology classes (real coefficients) of a real differentiable manifold M_n and regular exterior differential forms on M_n. This paper shows how the cohomology classes (integral coefficients, or integral coefficients mod p) can be represented by exterior forms. It is shown that if $\mathbf{C}^r$ is any cohomology class (integral coefficients), and ${}^*C^{n-r}$ is a cycle dual to $\mathbf{C}^r$ there exists a regular closed form Θ^r on M_n and a form Φ^{r-1} regular on $M_n - {}^*C^{n-r}$ such that (a) $\Theta^r = d\Phi^{r-1}$ on $M_n - {}^*C^{n-r}$; (b) $\int_z \Theta^r = \mathbf{C}^r \cdot \mathbf{Z}_r$, where z is a cycle of the arbitrary homology class $\mathbf{Z}_r$ (integral coefficients); (c) $\int_z \Theta^r - \int_{\partial z} \Phi^{r-1} = \mathbf{C}^r \cdot Z_r{}^{(p)}$ (mod p) where z is a cycle of the arbitrary homology class $\mathbf{Z}_r{}^{(p)}$ (integral coefficients mod p). Since an integral homology class is determined by its periods on all cycles mod p, the representation of integral homology classes by differential forms is achieved. Similar results are obtained for cohomology classes mod p, and for cochains with real coefficients.

De Rham's second theorem is also generalized by showing that if ω^r is any regular closed form on M_n, and ${}^*C^{n-r}$ is a cycle of the dual homology class, there exists a form Ω^{r-1} regular on $M_n - {}^*C^{n-r}$ such that, on $M_n - {}^*C^{n-r}$, $\omega^r = d\Omega^{r-1}$. When a Riemannian metric is given on M_n and harmonic forms are introduced, these results are given a more precise form in terms of Kodaira's integrals of the third kind.
W. V. D. Hodge (Cambridge, England).

Jongmans, François. Les diviseurs de zéro de l'anneau de cohomologie des variétés kählériennes. C. R. Acad. Sci. Paris **233**, 1254–1256 (1951).

In a Kähler manifold V_m denote by H the algebra (over the complex field) of harmonic forms. Two forms Φ_p, Ψ_q are called strict 0-divisors if $\Phi\cdot\Psi=0$, and (1) $p+q\leqq 2m$, (2) Φ, Ψ linearly independent, if $p=q=$odd ((1) and (2) exclude the trivial cases). Theorem: If the Betti numbers satisfy $b_p\neq 0$, $b_q\neq 0$, $b_{p+q}\leqq b_p+b_q-2$ (where $p, q>0, p+q\leqq 2m$), then there exist strict 0-divisors of degrees p and q; if $p=q=$odd one needs only $b_{2p}\leqq 2(b_p-2)$. For the proof one considers, roughly speaking, the dimension of the subvariety $H_p\cdot H_q$ of H_{p+q}. Using the relation $b_l\geqq b_{l-2}$ $(l\leqq m)$ for Kähler manifolds [Eckmann and Guggenheimer, same C. R. **229**, 503–505 (1949); these Rev. **11**, 212] finer results are established, in particular there are always strict 0-divisors, unless either $m=1$ or all $b_{2k}=1$ and all $b_{2k+1}=0$. *H. Samelson.*

Ehresmann, Charles. Les prolongements d'une variété différentiable. II. L'espace des jets d'ordre r de V_n dans V_m. C. R. Acad. Sci. Paris **233**, 777–779 (1951).

This note continues an earlier one [Ehresmann, same C. R. **233**, 598–600 (1951); these Rev. **13**, 386]; we indicate some definitions and theorems. If $\alpha(x)$, $\beta(x)$ are origin and endpoint of an r-jet, then α, β and $\gamma=(\alpha,\beta)$ are maps of the space of r-jets $J^r(V_n, V_m)$ into V_n, V_m, $V_n\times V_m$. By considering the extension to r-jets of the atlases defining V_n and V_m one shows: The maps α, β, γ are fiber maps, with fibers $T_n^r(V_m)$, $T_m^{r*}(V_n)$, $L^r_{m,n}$, and structure groups L_n^r, L_m^r, $L_n^r\times L_m^r$; the principal bundles can also be described. A cross-section of α is called an r-wave (r-flot) of V_n in V_m; a cross-section of β is an r-field (r-champ) of V_n in V_m. Differential forms of order r are defined as waves whose elements are p^r-covelocities. *H. Samelson.*

Ehresmann, Charles. Les prolongements d'une variété différentiable. III. Transitivité des prolongements. C. R. Acad. Sci. Paris **233**, 1081–1083 (1951).

This note continues that reviewed above. Let $0\leqq k\leqq r$; $l=r-k$. There is a natural map of $J^r(V_n, V_m)$ onto $J^k(V_n, V_m)$; the fiber structure of the latter over $V_n\times V_m$ is differentiable of class l. If F is a space on which L_n^k operates, one can define the extension (prolongement) $E(V_n, F)$; this is called regular if the operation of L_n^k is of class C^l. Theorem 1 states that an extension of order l of a regular extension of order k of V_n is an extension of order r ("transitivity"). Theorem 2: If L_n^k is transitive over F, then every extension of order l of $E(V_n, F)$ admits a subordinated fiber structure of base E, whose structure group is the isotropy group of L_n^r (operating on F by mapping it canonically on L_n^k).
H. Samelson (Ann Arbor, Mich.).

Guggenheimer, H. A note on curvature and Betti numbers. Proc. Amer. Math. Soc. **2**, 867–870 (1951).

S. Bochner [Ann. of Math. **49**, 379–390 (1948); **50**, 77–93 (1949); these Rev. **9**, 618; **10**, 571] has proved that if certain inequalities, depending on an integer k, $1\leqq k\leqq m$, are satisfied by the curvature tensor of a compact Kähler manifold of complex dimension m, then the Betti numbers R_p of the manifold satisfy the equations $R_{2l}=1$ $(2l\leqq k)$, $R_{2l+1}=0$ $(2l+1\leqq k)$. The object of the paper is to extend the proof to establish the formula $R_{2l}=1 \pmod 2$ $(l\leqq k)$. The proof is merely sketched and the reviewer has been unable to complete it.

In a concluding paragraph the author asserts that Bochner's results are valid for any compact Riemannian manifold of dimension $2m$ on which there exists a closed 2-form $\Omega=h_{ij}da^ida^j$ of rank 2ρ at all points of the manifold provided $2l$ (or $2l+1$)$\leqq\min(p,\rho)$. The definition of effective forms given in this paragraph is meaningless unless $\rho=m$, but the reviewer believes that the theorem can be established by a slight variation of the proof indicated, provided that h_{ij} has covariant derivative zero with respect to some Riemannian metric, and that this condition is necessary for the truth of the theorem. *W. V. D. Hodge.*

√★**Sokolnikoff, I. S. Tensor Analysis. Theory and Applications.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. ix+335 pp. $6.00.

The first part of this book gives an introduction to tensor theory. Chapter 1 contains a discussion in geometrical language of linear transformations in vector spaces and matrices of different kinds (symmetrical, orthogonal, hermitean, etc.). The second chapter is concerned with tensor algebra and tensor analysis in a differentiable manifold X_n. Tensors are introduced without making an appeal to geometrical interpretations. The definitions are similar to those given by Veblen and Whitehead. Covariant differentiation of tensors is defined only for Riemannian spaces. The remaining chapters deal with the applications of tensor calculus to geometry and mechanics. Chapter 3 gives an introduction to metric differential geometry. Apart from a few portions of Riemannian geometry it is mainly confined to the geometry of euclidean spaces (E_3), in which curvilinear coordinates have been introduced. It contains the principal theorems concerning space curves and surfaces. Chapter 4 contains applications to classical mechanics of particles and rigid bodies together with a deduction of the integral theorems of Gauss, Green and Stokes. After a brief introduction to relativistic mechanics the tensor calculus is applied to the mechanics of continuous media. The book is written for graduate students interested in applied mathematics which accounts for the elementary character of the exposition. The bibliography contains only texts written in English. *J. Haantjes* (Leiden).

NUMERICAL AND GRAPHICAL METHODS

★**von Sanden, H. Praktische Mathematik.** 2d ed. B. G. Teubner Verlagsgesellschaft, Leipzig, 1951. 120 pp.

In this terse little book is compressed a surprising amount of information. Chapter I, on the graphical treatment of functions, deals with the selection of a unit length, representation of a function by a curve, graphical representation of the definite integral and differential quotient, graphical multiplication, division and integration, use of the integraph, integration of products with strips of paper, the slide rule, logarithmic graph paper, and other forms of graph paper. In chapter II, on Taylor's theorem, the author considers numerical calculation and accuracy, replacement of a function by a polynomial, maxima and minima, Taylor's theorem for several variables, solution of equations,

Horner's method, and the method of iteration. In Chapter III, on integration, differentiation, and interpolation, he presents a few of the more important classical formulas, some in terms of differences and some in terms of the functional values themselves. Chapter IV, entitled Statistics, gives examples of distribution curves, the normal curve, the mean, the variance, and the use of probability paper. The method of Least Squares forms the subject of chapter V. In it are discussed the treatment of direct observations, the law of errors, weighted observations, indirect observations, nonlinear functions, and the smoothing of empirical functions. The final Chapter VI gives the procedure for harmonic analysis and synthesis by trigonometric interpolation using twenty four points. A considerable number of figures and numerical examples illustrate the text. *W. E. Milne.*

***Gumprecht, R. O., and Sliepcevich, C. M. Tables of light-scattering functions for spherical particles.** Engineering Research Institute, University of Michigan, Ann Arbor, Mich., 1951. xv+574 pp. $6.50.

Tables relating to diffraction of light by spherical particles have been available since 1949 [Tables of scattering functions for spherical particles, National Bureau of Standards. Appl. Math. Ser., no. 4 (1949); these Rev. **10**, 625]. However, for many purposes these tables are not extensive enough. The monumental tables under review cover a, roughly fifty-times, greater range of diameters, up to 127 wavelengths. Six (against four in the earlier tables) real indices of refraction were chosen for computation. On the other hand, values for absorbing materials and complex indices of refraction are not presented. In addition, the intensity functions are tabulated for an angle of observation of 90 degrees only. However, tables are included (Part II, see below) from which these functions for any angle can be obtained with comparative ease by using the tables reviewed in the next review but one. Definitions and notations as in the earlier tables [see the cited review]. Part I (6 pp.): Values of $K(m, \alpha)$; i_1 and i_2 for $\gamma=90°$; for $\alpha=1(1)6(2)$-$10(5)100(10)200(50)400$; $m=1.20$, 1.33, 1.40, 1.44, 1.50, 1.60 [4S]. Part II: Values of $\mathrm{R}(A_n)$, $\mathrm{I}(A_n)$, $\mathrm{R}(P_n)$, $\mathrm{I}(P_n)$ for the same range of α and m [6D, $n\leqq 421$].

The present tables were computed on the Eniac. "It is estimated that it would have taken one man, working 40 hours per week, with a standard desk computer, approximately 25 years to make similar computations. Less than two weeks' time was required once the details of the programming had been worked out." *C. J. Bouwkamp.*

***Gumprecht, R. O., and Sliepcevich, C. M. Tables of Riccati Bessel functions for large arguments and orders.** Engineering Research Institute, University of Michigan, Ann Arbor, Mich., 1951. xvi+260 pp. $3.50.

These tables are a by-product of the calculation of light-scattering functions for spherical particles [see the preceding review]. Ricatti Bessel functions can be defined in terms of ordinary Bessel functions by

$$S_n(x)=(\pi x/2)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x),\quad C_n(x)=(-1)^n(\pi x/2)^{\frac{1}{2}}J_{-n-\frac{1}{2}}(x).$$

Table I gives values of $S_n(x)$, $S_n'(x)$, $C_n(x)$, $C_n'(x)$ for

$$x=1(1)6(2)10(5)100(10)200(50)400;$$

$n=1(1)y$ where $y=x+\delta$, and δ gradually increases from 4 ($x=1$) to 20 ($x=400$) [at least 6D or 6S]. Table II contains $S_n(x)$ and $S_n'(x)$ for miscellaneous values of x ranging from $x=1.20$ to $x=640.00$, for the same values of n and to the same accuracy as in table I. *C. J. Bouwkamp.*

***Gumprecht, R. O., and Sliepcevich, C. M. Tables of functions of first and second partial derivatives of Legendre polynomials.** Engineering Research Institute, University of Michigan, Ann Arbor, Mich. 1951. xii +310 pp. $3.50.

Let $P_n(x)$ denote Legendre's polynomial, and let

$$\pi_n=P_n'(x),\quad \pi_n'=P_n''(x),$$

where a dash refers to differentiation with respect to x. The functions π_n and $x\pi_n-(1-x^2)\pi_n'$ occur in the problem of diffraction by spherical particles [see the second preceding review], where $x=\cos\gamma$ and γ is the angle of observation. These functions are tabulated (except for a factor 10^4) for $\gamma=0(10)170(1)180$ degrees and $n=1(1)420$. The tabulated functions are accurate to one unit in the fifth significant figure, and where fewer significant figures are given, to one unit in the last place given. Computations were done on an IBM Model 602-A Calculating Punch, resulting in 9 printed digits. [Reviewer's note: in the entries under $\gamma°$, pp. 119 and 121, the first digit 1 is missing.] *C. J. Bouwkamp.*

Emersleben, Otto. Numerische Werte des Fehlerintegrals für $\sqrt{n\pi}$. Z. Angew. Math. Mech. **31**, 393–394 (1951).

The values of $H(\sqrt{(n\pi)})$, where

$$H(x)=(2/\sqrt{\pi})\int_0^x \exp(-t^2)dt$$

are given for $n=1(1)10$. The value for $n=1$ has been computed to 17D from the power series. The values for $n=2(1)10$ are given to 15D and have been found by interpolation in National Bureau of Standards Tables of Probability Functions I, [New York, 1941; these Rev. **3**, 5]. For $n\geqq 11$, $H(\sqrt{(n\pi)})$ is unity, to 15D. The values are used in the computation of certain Epstein Zeta functions.

J. Todd (Washington, D. C.).

Bloch, I., Hull, M. H., Jr., Broyles, A. A., Bouricius, W. G., Freeman, B. E., and Breit, G. Methods of calculation of radial wave functions and new tables of Coulomb functions. Physical Rev. (2) **80**, 553–560 (1950).

The computation of Coulomb wave functions was started in 1946 under the auspices of the ONR. During the course of the work it has been found possible to employ some short cuts and methods of approximation which should be of help in the calculation of radial wave functions not only in the Coulomb case but also for more general central fields. A summary of these methods is presented in this paper. A characteristic feature of the problem is the fact that one needs two functions for each energy, the regular and the irregular functions. The regular solution can be started as a power series at $r=0$ and its calculation is easier than that of the irregular function. Means of making use of available relations between the two functions have been found. The present paper contains an introduction to the use of the new Coulomb functions tables which cover ranges of parameters required for the calculation of nuclear reactions of protons, deuterons and alpha-particles from hydrogen to oxygen in the energy range of a few Mev.

S. C. van Veen (Delft).

Uhler, Horace Scudder. Many-figure values of the logarithms of the year of destiny and other constants. Scripta Math. **17**, 202–208 (1951).

★von Neumann, John. The general and logical theory of automata. Cerebral Mechanisms in Behavior. The Hixon Symposium, pp. 1–31; discussion, pp. 32–41. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. $6.50.

This is a qualitative and stimulating discussion of certain general characteristics of computing automata and their analogies with the nervous systems of living organisms. The author stresses particularly the need of a logical theory of automata which will take into account the actual length of construction processes and allow for exceptions with low probabilities. He then discusses the McCulloch-Pitts theory of formal neural networks, and the Turing theory of idealized computing machines. At the end he outlines a generalization of the latter in which the possibility may be shown of constructing an automaton which is self-reproducing.

H. B. Curry (State College, Pa.).

★Krawitz, Eleanor. Matrix by vector multiplication on the IBM Type 602-A Calculating Punch. Proceedings, Industrial Computation Seminar, September 1950, pp. 66–70. International Business Machines Corp., New York, N. Y., 1951.

The process in the title is described in detail and the control panel wiring is given. *F. J. Murray.*

★Householder, Alston S. Analyzing exponential decay curves. Proceedings, Seminar on Scientific Computation, November, 1949, pp. 28–32. International Business Machines Corp., New York, N. Y., 1950.

The author surveys several numerical procedures for finding the best fit to given data by a sum of p exponentially decaying terms. The p decay constants may be found from the roots of a certain algebraic equation of degree p; the main problem is to estimate the coefficients of this equation. The Prony method for this estimation—the least-squares method described in some standard books—is criticized on theoretical grounds but is thought to be useful for obtaining first approximations which can be improved by iterative procedures. A modification of the Prony process, based on a matrix inversion method of Choleski, is suggested which may be especially useful in case p is unknown. A method is also presented which stems from a Poisson distribution of errors rather than from the least-squares approach.

P. W. Ketchum (Urbana, Ill.).

★Turner, L. Richard. Improvement in the convergence of methods of successive approximation. Proceedings, Computation Seminar, December 1949, pp. 135–137. International Business Machines Corp., New York, N. Y., 1951.

Expose le procédé bien connu pour améliorer la convergence d'un procédé d'itération lorsque les écarts sont en progression géométrique. *J. Kuntzmann* (Grenoble).

Križanovs'kiĭ, O. M. On the approximate determination of the roots, least in modulus, of numerical transcendental equations. Dopovidi Akad. Nauk Ukrain. RSR. **1950**, 11–14 (1950). (Ukrainian. Russian summary)

Let $P(x)=1+a_1x+a_2x^2+\cdots$ be a polynomial in x and a finite number of exponentials $\exp(\tau_i x)$, where a_i and τ_i are real (sometimes called a quasipolynomial). To determine the root of smallest modulus the author replaces $P(x)=0$ by $P_1(x)\equiv 1+(a_1-\alpha)x=0$; then formally

$$\alpha=\sum_{n=2}^{\infty}(-1)^k a_k(a_1-\alpha)^{-k+1},$$

and he apparently approximates α by truncating the series after 1, 2, 3, $\cdots$ terms, stating that this iterative process converges. To get the two roots of smallest modulus he uses $P_2(x)=1+(a_1-\alpha)x+(a_2-\beta)x^2$, determining α and β by iteration from a more complicated pair of simultaneous equations; and so on. The coefficients are written out also for the equations determining the three parameters in $P_3(x)$ and the four parameters in $P_4(x)$. The author works a numerical example using $P_4(x)$ to find the first four roots (two real and a complex pair). *R. P. Boas, Jr.* (Evanston, Ill.).

Bachmann, K.-H. Zur genäherten Auflösung algebraischer Gleichungen. Z. Angew. Math. Mech. **31**, 390–392 (1951).

Methods are described for the computation of approximations to the roots of algebraic equations with real coefficients and with no real roots. Methods are also shown for the computation of upper and lower bounds to the moduli of the roots and for the computation of the roots of algebraic equations with complex coefficients. *E. Frank.*

Grossman, D. P. On the problem of the numerical solution of systems of simultaneous linear algebraic equations. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 87–103 (1950). (Russian)

Some of the material contained in the paper by L. Fox, H. D. Huskey and J. H. Wilkinson [Quart. J. Mech. Appl. Math. **1**, 149–173 (1948); these Rev. **10**, 152] is here amplified and developed by one of the translators of that paper [cf. these Rev. **11**, 743]. (1) The Aitken "below the line" method is formulated concisely in terms of the coefficients of the original system of equations and its validity is proved by induction. A modification is explained which provides for the occurrence of certain matrix elements equal to zero (or nearly so) during the computation. To permit programming on standard punched-card equipment another slight modification is given which largely avoids division. This in turn is applied to evaluation of determinants. (2) Fox et al. referred briefly to a relation between the method of Morris and the method of orthogonal vectors explained in their paper. Here the exact extent of this equivalence is exhibited after a vector formulation of each method has been given which is not restricted [see Bodewig's review of the paper cited above] to systems with symmetric matrices. (3) An enumeration of operations is made. Over-all comparison leads the author to conclude that the three methods discussed here have no great advantage over ordinary Gauss elimination and back-substitution. *R. Church.*

Di Berardino, V., e Frandi, P. Formule ricorrenti per la risoluzione graduale dei sistemi di equazioni algebriche lineari. Archimede **2**, 108–113 (1950).

Die Verf. betrachten eine Kette von linearen Gleichungssysteme: $\mathfrak{A}^{(1)}X^{(1)}=\mathfrak{N}^{(1)}$; $\mathfrak{A}^{(2)}X^{(2)}=\mathfrak{N}^{(2)}$; $\mathfrak{A}^{(3)}X^{(3)}=\mathfrak{N}^{(3)}$; $\cdots$, wo

$$\mathfrak{A}^{(k)}=\begin{pmatrix} a_{11}\cdots a_{1k} \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ \cdot \qquad \cdot \\ a_{k1}\cdots a_{kk}\end{pmatrix}.$$

Aus (1) wird $X^{(1)}$ bestimmt, dann mittels $X^{(1)}$ und ziemlich einfach gebildete Koeffizienten-Komplexe die beiden Grössen $X^{(2)}$, darauf mittels die schon erhaltenen Grössen und neue Koeffizienten-Komplexe die drei Grössen $X^{(3)}$, u.s.w. Nach der Meinung der Verf. bietet diese Methode mehrere Vorteile gegenüber die sonst benützten Methoden, nämlich

grössere Einfachheit, grössere Schnelligkeit und grössere Genauigkeit (man vermeidet die Anhaufung der Rechenfehler). Die Arbeit enthält weiter ein Rechenschema und ein Anwendungsbeispiel auf einen Berechnung der Festigkeitslehre. *S. C. van Veen* (Delft).

Raymondi, Carlo. Contributo allo studio dei sistemi elastici staticamenta indeterminanti. Atti dell'Istituto di Scienza delle Costruzioni dell'Università di Pisa, Pubblicazione no. **13**, 1–15 (1949).

Di Berardino, Vincenzo, e Frandi, Paolo. Formule ricorrenti per la risoluzione graduale dei sistemi di equazioni algebriche lineari. Ricerca Sci. **20**, 662–666 (1950).

The first paper solves n linear equations by the escalator process: the column of knowns is written to the right of the nonsymmetric square matrix of coefficients and at the kth stage of the computation k elements of the $(k+1)$st column and k elements of the kth row are brought in. The method of the second paper is equivalent to applying identical steps to the n by $n+1$ matrix obtained by writing the knowns of the system diagonally just to the right of the main diagonal coefficients so that here (without obvious advantage) the knowns enter the computation from the start. In each case $(2n^3+3n^2-5n)/6$ additions-subtractions, $(n^3+3n^2-n)/3$ multiplications and n reciprocals are needed. Both papers are strongly motivated by interpretations in the statics of structures. The second is a development of earlier work (1935) in this direction by Di Berardino while the other is influenced by F. Jossa who was led [Rend. Accad. Sci. Fis. Mat. Napoli (4) **10**, 346–352 (1940); these Rev. **8**, 535] to the equations for finding the inverse of a matrix by bordering. *R. Church* (Monterey, Calif.).

***Luckey, Bonalyn A. Inversion of an alternant matrix.** Proceedings, Computation Seminar, December 1949, pp. 43–46. International Business Machines Corp., New York, N. Y., 1951.

The inverse of the alternant or Van der Monde determinant is described. A special method of evaluating the terms is advocated for punched card equipment.
F. J. Murray (New York, N. Y.).

***Chancellor, Justus, Sheldon, John W., and Tatum, G. Liston. The solution of simultaneous linear equations using the IBM Card-Programmed Electronic Calculator.** Proceedings, Industrial Computation Seminar, September 1950, pp. 57–61. International Business Machines Corp., New York, N. Y., 1951.

The solution process involving the customary diagonalizing and "back solution" is described for the CPC with checks. The matrix is not "conditioned" for this process.
F. J. Murray (New York, N. Y.).

Roberson, R. E., and McCool, W. A note on the analog computation of determinants. NRL Rep. **3557**, Naval Research Laboratory, Washington, D. C., iv+8 pp. (1949).

An interim report on a method for using a Reeves Electronic Analog Computer for evaluating determinants by solving for selected variables in a succession of linear algebraic equations. A procedure given by W. McCool is used to obtain stable solutions. The size of the determinant which can be handled is limited by the number of adding circuits available in the computer. No estimate of accuracy is given.
S. H. Caldwell (Cambridge, Mass.).

Bückner, Hans. Bemerkungen zur numerischen Quadratur. I. Math. Nachr. **3**, 142–145 (1950).

Es wird gezeigt, dass es nicht möglich ist ein System von Abszissen

$$(1)\qquad 0\leqq x_1^{(n)}<x_2^{(n)}<\cdots<x_n^{(n)}\leqq 1$$

und von zugehörigen Gewichten (2) $g_m^{(n)}\geqq 0$ $(m=1,2,\cdots,n)$ so zu bestimmen, dass der Fehler

$$r_n(f)=\sum_{i=1}^{n} g_i^{(n)} f(x_i^{(n)})-\int_0^1 f(x)dx$$

zugleich für alle ρ-mal stetig differenzierbaren Funktionen $f(x)$ stärker als mit $n^{-\rho}$ gegen Null konvergiert für $n\to\infty$. Genauer wird gezeigt: Es gehöre $f(x)$ zur Menge $\mathfrak{M}$ aller ρ-mal stetig differenzierbaren Funktionen. Die Abszissen (1) und Gewichte (2) seien für alle natürlichen n definiert. Unabhängig davon sei eine beliebige positive Funktion $\varphi(n)$ erklärt, von der nur verlangt wird, dass sie mit n über alle Grenzen wächst. Dann existiert mindestens eine Funktion $f(x)\subset\mathfrak{M}$, die ihrerseits noch von φ abhängen kann, so dass die Aussage $\lim_{n\to\infty} r_n(f)\varphi(n)n^\rho=0$ falsch ist. Beispiel:

$$H(x)=x^{\rho+1}(1-x)^{\rho+1};\quad \int^1 H(x)dx=C;$$
$$\max_{0\leqq x\leqq 1}|H^{(\nu)}(x)|=C_\nu,\quad \nu=0,1,\cdots,$$
$$H_n(x)=hH(t)\quad\text{mit}\quad t=(x-x_k^{(n)})/(x_{k+1}^{(n)}-x_k^{(n)})$$

und

$$h=n^\rho(x_{k+1}^{(n)}-x_k^{(n)})^\rho,\quad x_k^{(n)}\leqq x\leqq x_{k+1}^{(n)}\quad (k=0,1,\cdots,n),$$
$$x_0^{(n)}=0,\quad x_{n+1}^{(n)}=1.$$

Die Funktion $\varphi(n)$ ist eine für alle natürlichen n definierte positive und mit n über alle Grenzen wachsende Funktion. Es wird gezeigt, dass die natürliche Zahlen $n_1, n_2, \cdots$ immer so bestimmt werden können, dass die Funktion $F(x)=\sum_{k=1}^\infty H_{n_k}(x)n_k^{-\rho}/\varphi(n_k)$ das Gefragte leistet.
S. C. van Veen (Delft).

Bückner, Hans. Bemerkungen zur numerischen Quadratur. II. Math. Nachr. **3**, 146–151 (1950).

In Anschluss an die vorhergehende Arbeit wird gezeigt, dass man die Konvergenz $\lim_{n\to\infty} r_n(f)n^\rho=0$ bereits mit äquidistanten Abszissen (1) und mit solchen Gewichten (2) erreichen kann, von denen höchstens 2ρ sich von $1/n$ unterscheiden. Beispiel: Die Funktion $f(t)=H(s,t)$ ist $(\rho-2)$-fach stetig differenzierbar nach t in $0\leqq s\leqq 1$, $0\leqq t\leqq 1$. Die $(\rho-1)$te und ρte Ableitungen existieren in $0\leqq t\leqq s$, $s\leqq t\leqq 1$ und sind stetig. Die Laplacesche Integrationsformel

$$\int_0^1 H(s,t)dt\approx\sum_{i=1}^{n}\frac{1}{n}H\left(s,\frac{i-\frac12}{n}\right)$$
$$+\sum_{\nu=1}^{\rho-1}A_\nu\left\{\Delta^\nu\left[H\left(s,\frac{i-\frac12}{n}\right)+H\left(s,\frac{n-i+\frac12}{n}\right)\right]\right\}$$
$$=\sum_{i=1}^{n} g_{\rho i}^{(n)} H\left(s,\frac{i-\frac12}{n}\right)\frac{1}{n}$$

(wo die Grössen A_ν bestimmte Zahlenkoeffizienten bedeuten) liefert, wenn $D_{\rho-1}H(s,t)$ stetig ist für $0<s<1$,

$$\lim_{n\to\infty} n^\rho\left\{\int_0^1 H(s,t)dt-\sum_{i=1}^{n} g_{\rho i}^{(n)} H\left(s,\frac{i-\frac12}{n}\right)\frac{1}{n}\right\}=0$$

und zwar gleichmässig in ρ. Dieses Ergebnis wird erweitert auf den Fall, dass $D_{\rho-1}H(s,t)$ unstetig wird im Gebiet $0<s<1$. *S. C. van Veen* (Delft).

Mineur, Henri. Tentatives de calcul numérique des intégrales doubles. C. R. Acad. Sci. Paris **233**, 1166–1168 (1951).

The author develops a theory of numerical cubature which is analogous to the quadrature method of Gauss. The problem is to determine a minimum set of points in the square $-1 \leq x, y \leq 1$ such that the integral over this square of a polynomial of degree n shall be given exactly by a fixed linear combination of values of the integrand at these points. A difficulty arises in that the natural analogy to the Gauss method gives a system of equations which is incompatible. Hence it is necessary to use more points than is required to give an equal number of unknowns as equations. Numerical values for the points and coefficients are given for $n=5$ and 7. When applied to a certain exponential integral, the case $n=7$ yielded an accuracy of 1 part in 10000.

P. W. Ketchum (Urbana, Ill.).

Tortorici, Paolo. Su un metodo numerico di calcolo approssimato per gli integrali doppi. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **303**, 7 pp. (1951).

Let (a, b) be interior to the rectangular domain

$$R[a' \leq x \leq a'', b' \leq y \leq b''].$$

For $k=1, 2$, let $u^{(k)}(x, y) = \partial^{2k}(x, y)/\partial x^k \partial y^k$, called the "total derivative of order k." It is proved under appropriate continuity conditions that

$$(*) \quad u(x, y) = u(a, y) + u(x, b) - u(a, b) + \int_a^x \int_b^y u^{(1)}(\xi, \zeta) d\xi d\zeta.$$

Then (*) is integrated over the rectangle

$$R_1[|a-x| \leq \alpha, |b-y| \leq \beta],$$

to approximate the integral $I = \iint_{R_1} u(x, y) dx dy$. One finds

$$(**) \quad I = 2\alpha \int_{b-\beta}^{b+\beta} u(a, y) dy + 2\beta \int_{a-\alpha}^{a+\alpha} u(x, b) dx - 4\alpha\beta u(a, b) + \rho.$$

If ρ is ignored, (**) requires the knowledge of u only along the lines $x=a$ and $y=b$, and the integrals are one-dimensional. From (*) one learns that

$$\rho = \int\int_{R_1} dx dy \int_a^x \int_b^y u^{(1)}(\xi, \zeta) d\xi d\zeta.$$

Analogous to (*), there is a formula giving $u(x, y)$ in terms of $u^{(1)}(x, y)$; it leads to an alternate expression for ρ in (**) as an integral involving $u^{(2)}(x, y)$.

If R_1 is divided into m^2 congruent rectangles, the application of (**) to each rectangle (ignoring ρ) yields useful numerical quadrature formulas for double integrals, even for small m. Bounds are given for the errors. As an example, $\int_0^1 dx \int_0^1 \log(1+x+y) dy \doteq 0.67116$ is evaluated with $m=2$ to an accuracy of 0.00006.

The methods will apply in more dimensions, and to regions transformable to rectangular ones. [Reviewer's note: The advantage of integrating (*) instead of integrating Taylor's formula seems to lie in the relative simplicity of the error term ρ.] *G. E. Forsythe* (Los Angeles, Calif.).

Birindelli, Carlo. Sul calcolo numerico degli integrali multipli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 40–44 (1951).

The idea is to approximate r-fold multiple integrals over a rectangular solid R in r dimensions in terms of $(r-1)$-fold multiple integrals over $(r-1)$-dimensional interior flats parallel to the faces of R, with an error easily majorized. Using the "total derivative" of order k,

$$u^{(k)}(x_1, \cdots, x_r) = \partial^{rk} u / \prod_i \partial x_i^k,$$

the author generalizes all the formulas of the preceding review from $r=2$ to any integer r, and from $k=1$ or 2 to any integer k. The formulas are too complicated to reproduce. The author notes that, by successive reduction, one could express r-fold multiple integrals in terms of certain values of the integrand, with a majorized error.

G. E. Forsythe (Los Angeles, Calif.).

Friedman, Bernard. Numerical methods for evaluation of the integrals for virtual height. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-17**, ii+34 pp. (1950).

The author obtains the virtual height of a given ionosphere layer as a function of the frequency by means of certain integrals involving a Chapman ionization distribution, which are obtained from the Appleton-Hartree formula. It is assumed that the number N of ions in the atmosphere varies with the height according to the Chapman distribution

$$N = N_0 \operatorname{Ch}(x) = N_0 \exp\left[\frac{1-x-e^{-x}}{2}\right]$$

where N_0 is the maximum ion density of the layer, x is the height above the earth in scale units defined by $x=(h-h_m)/H$, h_m = height of maximum ionization, h = actual height above the earth, $H = kT/mg$ = "scale height" [Chapman, Proc. Phys. Soc., London **43**, 26–45, 483–501 (1931)]. The integrals required for computing the virtual height are transformed to the standard form

$$I = \int_{-\infty}^{x_0} \frac{f(x) \operatorname{Ch}(x) dx}{[1 - \operatorname{Ch}(x)/\operatorname{Ch}(x_0)]^{\frac{1}{2}}}$$

where x_0 is the appropriate reflection height and $f(x)$ takes a particular form for each condition of propagation, e.g. $f(x) = \{\operatorname{Ch}(x_0)(1 + [1 - \operatorname{Ch}(x)/\operatorname{Ch}(x_0)]^{\frac{1}{2}})\}^{-1}$. These integrals are so complicated that it is impossible to evaluate them analytically. It is proposed by the author to apply the Gauss-Christoffel formula for the numerical evaluation of these integrals. It is shown that the Chapman distribution may be approximated very closely by a cosine curve $\frac{1}{2}(1+\cos(\pi x/2x'))$, $0 > x > -2x'$, where x' is the abscissa of the point of inflection of the Chapman distribution. It is indicated how the calculated values of the virtual height may be used to analyse experimental data.

S. C. van Veen (Delft).

Aprile, Giuseppe. Calcolo grafico di integrali di Stieltjes, mediante poligoni funicolari. Giorn. Sci. Nat. Econ. Sez. I. **45**, no. 8, 2 pp. (1948).

Die Wert des Stieltjesintegrales

$$S(t) = \int_{x_0}^t A(x) dB(x)$$

wird angenähert bestimmt mittels der Summationsformel:

$$(1) \qquad \sum_{k=0}^{n} A(k\epsilon) D(k\epsilon)\epsilon$$

wo $D(x) = dB(x)/dx$, $x_0=0$, $t=n\epsilon$ (n ganz). Die Grösse $D(k\epsilon)\epsilon$ wird betrachtet als eine Parallelkraft F_k, und die Grösse $A(k\epsilon)$ als der Distanz zur x-Achse. Die Summe (1) wird nun grafisch bestimmt mittels ein Seilpolygon.

S. C. van Veen (Delft).

Haag, Rudolf. Über eine Methode der Störungsrechnung und ihre Anwendung auf Schwingungsprobleme. Z. Angew. Math. Mech. 31, 12–19 (1951). (German. English, French, and Russian summaries)

The author's starting point is the differential equation for the forced vibration of the linear oscillator in which the input consists of a weighted sum of simple harmonics, and which is given by

$$(1)\qquad \ddot{x}+\beta\dot{x}+\omega^2x=\sum K_k\cos(w_kt-\phi_k).$$

Under well known conditions the solution of (1) is a linear aggregate of damped oscillations with respective frequencies $\omega\pm w_k=\Omega_k$. The author then introduces 'perturbations' in equation (1) which consist of the addition of the terms $(\alpha\cos pt)x+\lambda x^2$. He discusses the solution of the 'perturbed' equation (1) for small λ and/or α. A solution in the form

$$x=\sum_k A_k(\alpha,\lambda)\exp\{i\Omega_k(\alpha,\lambda)t\}$$

is attempted in which both Ω_k as well as A_k are expanded in powers of α and λ. Starting from $\Omega_k(0,0)=\Omega_k$, recurrent equating of coefficients of powers of $\alpha^n\lambda^m$ yields the initial terms of these expansions. The linear equation ($\lambda=0$) is considered in detail whilst for $\lambda\neq0$ the method is only workable for oscillatory solutions. *H. O. Hartley* (London).

Salzer, Herbert E. Formulas for numerical integration of first and second order differential equations in the complex plane. J. Math. Physics 29, 207–216 (1950).

In calculations with analytic functions in the complex plane, it is generally more natural and expedient to retain z as the variable, rather than x and y. The author has derived formulas for complex integration and interpolation, based upon approximation of the function to be integrated or interpolated as a polynomial in z, namely the Lagrangian interpolation polynomial. These formulas are for complex integration in a cartesian grid, based upon three to nine points (i.e. assuming the integrand to behave as a polynomial in z which may range from the second through the eighth degree in the vicinity of the fixed-points). The choice of the configurations of the fixed points z_j was influenced by three considerations: (I) Convenience of an L-shaped corner, which is also suggestive of initiation of the integration in either the x- or y-direction, beginning from a corner of the desired region in the z-plane. (II) The points z_j should be as close to each other as possible. Thus in the five- and seven-point cases the points z_j were not chosen to lie symmetrically about the 45° ray even though that would have meant a lessening in the labor of deriving all the integration formulas in order to obtain that slight theoretical advantage of closer points. (III) The configuration should be such that as little mental effort as possible is required to picture it moved one space to the right, or one space upward, by extrapolation. *S. C. van Veen* (Delft).

Stöhr, Alfred. Zur approximativen Lösung linearer homogener Differentialgleichungssysteme. Math. Nachr. 6, 97–102 (1951).

The author is concerned with the initial value problem of a system of j linear first order differential equations which he writes in the Stieltjes integrated form

$$(1)\qquad u_k(x)=v_{k,0}+\sum_{l=1}^{j}\int_a^x u_l(\xi)d\phi_{kl}(\xi),\quad k=1,2,\cdots,j,$$

and for which the solutions $u_k(x)$ with $u_k(a)=v_{k,0}$ have to be found for $a\leqq x\leqq b$. Assuming that the $\phi_{kl}(x)$ have a bounded total variation over the interval a, b, it is reproved that the system (1) has only one set of bounded solutions consisting of a set of continuous functions $u_k(x)$. It is further shown how a variety of approximate solutions can be constructed by Stieltjes integration of step functions satisfying a proximity condition to the $\phi_{kl}(x)$. For all these approximations a common gauge of accuracy is derived which depends on the $v_{k,0}$ and on the total variation of the $\phi_{kl}(x)$. *H. O. Hartley* (London).

√ ***Milne, W. E. Numerical methods associated with Laplace's equation.** Proceedings of a Second Symposium on Large Scale Digital Calculating Machinery, 1949, pp. 152–163. Harvard University Press, Cambridge, Mass., 1951. $8.00.

L'auteur passe en revue les principales difficultés qui se présentent dans la solution approchée des équations aux dérivées partielles par des méthodes de différences, lorsqu'on utilise de grandes machines à calculer. Discussion des diverses formes de mailles. Résolution du système d'équations du 1er degré obtenu par approximations successives. Examen des formules du type

$$V_{n-1}=V_n+\frac{1}{\theta}\begin{bmatrix}1&4&1\\4&-20&4\\1&4&1\end{bmatrix}V_n$$

pour l'équation de Laplace. Discussion suivant les valeurs de θ. Lorsque le nombre de points est très grand, la convergence est très lente. Il y aurait intérêt à mettre au point des méthodes analogues à celle de Southwell et s'appliquant aux machines. Indications sommaires sur diverses autres difficultés. *J. Kuntzmann* (Grenoble).

√ ***Lanczos, Cornelius. An iteration method for the solution of the eigenvalue problem of linear differential and integral operators.** Proceedings of a Second Symposium on Large-Scale Digital Calculating Machinery, 1949, pp. 164–206. Harvard University Press, Cambridge, Mass., 1951. $8.00.

Vergleiche eine schon referierte Arbeit des Verfassers [J. Research Nat. Bur. Standards 45, 255–282 (1950); diese Rev. 13, 163]. *P. Funk* (Wien).

***Flanders, Donald A., and Shortley, George H. Eigenvalue problems related to the Laplace operator.** Proceedings, Seminar on Scientific Computation, November, 1949, pp. 64–70. International Business Machines Corp., New York, N. Y., 1950.

In dieser Arbeit berichten die Verfasser über die selben Gedanken, die sie in ihrer früheren Arbeit [J. Appl. Phys. 21, 1326–1332 (1950); diese Rev. 12, 640] dargelegt haben. Wieder werden die Probleme auf algebraische Probleme reduziert, wo sämtliche Eigenwerte im Intervall $(-1,1)$ liegen. Bevor sie wieder auf die Verwendung von Tschebyscheff-Polynomen eingehen, werden zunächst lineare Operatoren besprochen und geeignete Produktbildungen von linearen Operatoren. Es finden sich zahlreiche praktische Hinweise, wie man in Einzelfällen vorzugehen hat, so z.B. wird der Fall genauer besprochen, dass λ_1 und λ_2 wenig voneinander verschieden sind. *P. Funk* (Wien).

***Hummel, Harry E. An eigenvalue problem of the Laplace operator.** Proceedings, Computation Seminar, December 1949, pp. 29–34. International Business Machines Corp., New York, N. Y., 1951.

Im Anschluss an das Verfahren von Flanders und Shortley [siehe das vorstehende Referat und die da angeführte Ab-

handlung] werden Einzelheiten der Durchführung der Rechnung mittels IBM Maschinen dargelegt. Aus der darauffolgenden Discussion, soweit sie die Arbeiten von Flanders und Shortley betrifft sei hier eine Bemerkung von Kunz hervorgehoben, dass mit der Steigerung der Genauigkeit eine Verbesserung der Annäherung an der Laplaceschen Ausdruck verbunden sein muss und dass es nicht nur auf die Anzahl der Maschen und Iterationen ankommt. In verschiedenen Bemerkungen wird insbesondere auf eine Arbeit von Aitken [Proc. Roy. Soc. Edinburgh 57, 172–181, 269–304 (1937)] hingewiesen. *P. Funk* (Wien).

Donsker, M. D., and Kac, M. A sampling method for determining the lowest eigenvalue and the principal eigenfunction of Schrödinger's equation. J. Research Nat. Bur. Standards **44**, 551–557 (1950).

The eigenvalues λ_j and the corresponding eigenfunctions of the one-dimensional Schrödinger equation

$$\frac{1}{2}\frac{d^2\psi}{dx^2}-V(x)\psi(x)=-\lambda\psi,\quad \lambda>0,$$

are found by a method utilising various probability techniques and sampling methods. These methods have become known as "the Monte-Carlo method". $X_1, X_2, X_3, \cdots$ are independent identically distributed random variables, each having mean 0 and standard deviation 1. Let $S_k=\sum_1^k X_n$. Then $\sigma(\alpha, t)$ is the limiting distribution function of the random variable $n^{-1}\sum_{k<nt} V(S_k/n^{1/2})$. It is shown that the lowest eigenvalue is:

$$\lambda_1=\lim_{t\to\infty}\frac{-1}{t}\log\int_0^\infty e^{-\alpha}d_\alpha\sigma(\alpha, t).$$

A more practical approximation is:

$$\lambda_1\sim\frac{1}{t_2-t_1}\log\frac{\int_0^\infty e^{-\alpha}d_\alpha\sigma(\alpha, t_1)}{\int_0^\infty e^{-\alpha}d_\alpha\sigma(\alpha, t_2)}.$$

The Monte Carlo process consists in the calculation of $\sigma(\alpha, t_1)$, and $\sigma(\alpha, t_2)$ by a sampling process. The authors consider two examples: $V(x)=x^2$ and $V(x)=|x|$. For these n is selected to be 400, $t_1=5$, $t_2=3.75$; 100 samples were used. For λ_1, the agreement with the exact value is rather good. The second eigenvalue is also calculated. Here the agreement is poor. *S. C. van Veen* (Delft).

***Donsker, M. D., and Kac, Mark. The Monte Carlo method and its applications.** Proceedings, Computation Seminar, December 1949, pp. 74–81. International Business Machines Corp., New York, N. Y., 1951.

This paper is essentially the same as the one reviewed above.

Allen, D. N. de G., and Dennis, S. C. R. Graded nets in harmonic and biharmonic relaxation. Quart. J. Mech. Appl. Math. 4, 439–443 (1951).

Formulas are given for use in going from a coarse to a finer network in the solution of the biharmonic equation by relaxation methods. An intermediate "diagonal" net provides the transition from the coarse to the finer region. This diagonal net is the same as that customarily employed for a transition region in the solution of the harmonic equation by relaxation methods. The formulas are derived by considering the biharmonic operator as the application of the harmonic operator twice. Residuals in the transition region are found to depend on values of the desired function in the coarse, transition, and fine mesh regions. *S. Levy.*

Wood, R. H. A special type of group displacement for use in the relaxation technique. Quart. J. Mech. Appl. Math. 4, 432–438 (1951).

A special type of group displacement is presented which can be used to advantage in the relaxation technique. In this technique, group displacements at several neighboring points are combined in such a way that their net effect is to cause a large change in a central point, zero change in immediately neighboring points, and a moderate change in an outer ring of points. Thus a large residual at a particular point can be spread to outlying points without introducing new residuals at immediately neighboring points. Patterns for use with both the harmonic and biharmonic equations are given. The effects of edges and corners are considered. The author recommends use of the largest possible pattern within the given boundaries to transfer as much of the residual as possible over the boundary. A numerical table shows the over-relaxation required in the case of the biharmonic equation. *S. Levy* (Washington, D. C.).

Space simulator. United States Atomic Energy Commission, Rep. **AECD-2298,** ii+37 pp. (1949).

A theoretical and experimental study of methods for using both active and passive electric network analogs to obtain solutions of simultaneous partial differential equations of the type

$$\nabla^2\phi-a\phi+b\theta=0,\quad \nabla^2\theta-c\theta+d\phi=0.$$

While the discussion emphasizes the treatment of these particular equations, it is broad enough to serve as a useful guide in extending the network method to the solution of other problems involving partial differential equations. The theoretical section derives the basic relationships between circuit parameters and equation coefficients. Factors governing the choice of mesh interval and transitions from coarse to fine meshes are treated. Extensions of the theory consider the problems of three-dimensional networks and the solution of n simultaneous equations. Both active and passive networks are described and their advantages and disadvantages are discussed at length. From experimental studies based on a one-dimensional model the origins and magnitudes of error are analyzed and discussed in detail. Experimental results were found to be accurate to about 2 per cent. *S. H. Caldwell* (Cambridge, Mass.).

Garwin, R. L. A differential analyzer for the Schrödinger equation. Rev. Sci. Instruments 21, 411–416 (1950).

The computing aspects of this paper are limited largely to a consideration of the Schrödinger equation. However, it contains considerable practical information regarding the circuits and problems encountered in building feedback-type electronic integrators. *S. H. Caldwell.*

Cutkosky, R. E. A Monte Carlo method for solving a class of integral equations. J. Research Nat. Bur. Standards **47,** 113–115 (1951).

L'auteur donne un procédé basé sur les probabilités pour trouver des valeurs approchies de $\varphi(x_k)$, x_k donné, $\varphi(x)$ étant solution de l'équation intégrale

$$\varphi(x)=f(x)+\lambda\int_a^b K(x, y)\varphi(y)dy.$$

Un procédé basé sur le même principe avait déja été donné pour l'inversion des matrices par Forsythe et Leibler [Math. Tables and Other Aids to Computation **4**, 127–129 (1950); ces Rev. **12**, 361]. *J. Kuntsmann* (Grenoble).

Cochran, W. The Fourier method of crystal-structure analysis. Acta Cryst. **1**, 138–142 (1948).

The purpose of this note is to investigate the relation between the Fourier method and other methods to derive accurate atomic coordinates from the data provided by an X-ray investigation of a crystal structure. The atomic coordinates are defined as those points at which the electron density reaches a maximum and around which the electron distribution is spherically symmetric in a small region. It is shown that the customary Fourier series method is closely related to the least-squares method applied to this problem by Hughes [J. Amer. Chem. Soc. **63**, 1737–1752 (1941)]. On the basis of this result a way in which the Fourier method can be modified to give less weight to certain coefficients has been discovered. The effect of terminating the Fourier series or introducing an artificial temperature factor is also investigated from the point of view of the relation between the Fourier and least squares solutions. The conclusion is that no significant increase in accuracy can be obtained by abandoning the Fourier method which presents many advantages ((a) within limits no assumption regarding the electron distribution in atoms need be made; (b) the relative values of the atomic scattering factors need be known only with sufficient accuracy to enable one to calculate correctly which sign is to be associated with each coordinate; and (c) the computations involved lend themselves more readily to mechanisation). The disadvantages of the unmodified Fourier method (e.g. no account is taken of the fact that some observations are much less accurate than others, and errors are introduced by the termination of the series) can be at least partially eliminated. *S. C. van Veen.*

Qurashi, M. M. Optimum conditions for convergence of steepest descents as applied to structure determination. Acta Cryst. **2**, 404–409 (1949).

The method of steepest descents consists in first forming the residual

$$R=\sum_{hkl} W^2(\phi_0-\phi_c)^2$$

where the ϕ's represent any single-valued, differentiable function of the atomic parameters $x_1, x_2, \cdots, x_i, \cdots, x_n$. (The parameters are not necessarily restricted to positive co-ordinates.) ϕ_0 is the observed value of ϕ and ϕ_c the corresponding calculated value, while W^2 is the weight allotted to any one value of $(\phi_0-\phi_c)^2$. The best values of the parameters are to be obtained by minimising R. R, being a function of the n variables x_i, can be represented by constant-R surfaces in n-dimensional space. The essence of the method is that successive approximations are obtained by moving along the normal to the R contours towards lower values of R. The successive approximations converge to the true values of the parameters at a rate, which depends very markedly on the scales of representation chosen for these parameters. The importance of this is discussed in detail and transformations are derived which secure the optimum rate of convergence and at the same time make all the parameters mathematically equivalent. The application of these transformations leads to the simple formula

$$\epsilon_i=\sum_{hkl} W^2(\phi_0-\phi_c)\frac{\partial\phi_c}{\partial x_i}\Big/\sum_{hkl} W^2\left(\frac{\partial\phi_c}{\partial x_i}\right)^2$$

for the corrections ϵ_i. An illustration of the use of this formula is given from the derivation of the structure of the hexagonal S-phase in the Ag-Zn alloy system. *S. C. van Veen* (Delft).

Cruickshank, D. W. J. The convergence of the least-squares and Fourier refinement methods. Acta Cryst. **3**, 10–13 (1950).

The purpose of the author is to extend the results of Qurashi [cf. the paper reviewed above] by examining the connexion between the corrections given by the least-squares (or improved steepest-descents) and Fourier methods. A direct connexion is established between the two methods for the final stages of refinement, though this takes slightly different forms for centrosymmetric and non-centrosymmetric structures, which are considered separately. A modified form of the Fourier method is suggested to allow for phase-angle refinement in non-centrosymmetric structures, which is equivalent to the least-squares method when the corrections are small. *S. C. van Veen* (Delft).

Vand, Vladimir. A simplified method of steepest descents. Acta Cryst. **4**, 285–286 (1951).

Booth's method of steepest descents, for refinement of crystal structure coordinates, is modified so that the residual function is separated into two terms. One term is a function of the observed structure factor only; the other term is in the form of a Fourier synthesis in which the coefficients change in sign but have absolute values which are independent of the observed and calculated structure factors. It is believed that the suggested modification will facilitate the use of automatic computing equipment in this type of calculation. *S. H. Caldwell* (Cambridge, Mass.).

Vand, V. A mechanical X-ray structure-factor calculating machine. J. Sci. Instruments **27**, 257–261 (1950).

Description of an analog computer of the tide-prediction type for performing X-ray crystallographic structure-factor calculations for up to 24 atoms. It computes at a speed of about 8 structure-factors per minute. Facilities for refining the atomic coordinates by the method of steepest descents are included but the operation of this feature is not entirely satisfactory. *S. H. Caldwell* (Cambridge, Mass.).

Robertson, J. H. A simple machine capable of Fourier synthesis calculation. J. Sci. Instruments **27**, 276–278 (1950).

A low-cost analog computer of reasonable accuracy. Sinusoidal motions of controllable amplitudes and frequencies are set up by a system of wheels and strings. These motions are transmitted to a system of levers which control the deflections of a group of mirrors. The motion of an output spot of light is proportional to the algebraic sum of the motions of the mirrors, so that the output represents the summation required in the Fourier synthesis.
S. H. Caldwell (Cambridge, Mass.).

Ramsayer, Karl. Funktionsrechenmaschinen mit ein- und mehrstufiger Interpolation. Z. Angew. Math. Mech. **31**, 301–309 (1951).

The author describes a method of mechanical storage of a function $y(x)$ in calculating machines in the form of special, optimum interval interpolation tables using a set of arguments x_i. The methods of interpolation advocated are not standard. For example, the linear interpolation formula used for $x_i\leqq x\leqq x_{i+1}$ is of the form (1) $a_i+b_i(x-x_i)$ with $a_i\neq y(x_i)$

and both a_i and b_i suitably chosen. Similar formulae are used for high order interpolation and their remainder terms are discussed in detail. Of the mechanical storage devices described one (apparently intended as an attachment to a barrel machine of the Brunsviga type) is sketched in more detail. It consists of metal plate stencils, rotably assembled on an axle, on which the digits of the a_i and b_i are represented as cuts of varying depth. Racks engaging the register wheels are pulled and arrested by these cuts thereby transferring a_i into the Product Register and b_i onto the Ohdner wheels of the multiplicand register. The multiplication (1) is then to be performed. No reference to an actually constructed device is given. The merits of storing $y(x)$ as an interpolation table are compared with those of other methods of storage. *H. O. Hartley* (London).

Bayly, J. G. An analog computer. Rev. Sci. Instruments **21**, 228–231 (1950).

An analog computer is described which is primarily adapted for the computation of yields from nuclear reactions, using either recorded data or using signals obtained directly from an instrument such as an ionization chamber. The principle feature is a servo type of integrator which is discussed in detail. *S. H. Caldwell* (Cambridge, Mass.).

Gray, E. P., and Follin, J. W., Jr. First report on the REAC computer. Applied Physics Laboratory, The Johns Hopkins University, Silver Spring, Md., APL/JHU TG-50, 33 pp. (1948).

Although this is an interim report on the use of the REAC it contains two sections of particular value. Section II discusses circuits for the solution of typical problems and contains descriptions of a number of basic techniques. Section III is an analysis of the errors in machine operation which can be of considerable guidance to users. It is pointed out that in the computation of trajectories or in the evaluation of definite integrals the machine can introduce substantial error, but that the effect of the error sources is greatly reduced in computing, for example, solutions which are of an oscillatory type. *S. H. Caldwell.*

✶Liggett, Irving C. Two applications of the IBM Card-Programmed Electronic Calculator. Proceedings, Industrial Computation Seminar, September 1950, pp. 62–65. International Business Machines Corp., New York, N. Y., 1951.

The Gauss-Seidel method and Newton's root finding method are described for the C. P. C. *F. J. Murray.*

Mitrovic, Dusan, Huron, Roger, et Tomovic, Rajko. Sur un principe nouveau de construction des machines servant à résoudre les systèmes d'équations linéaires par analogie électrique. C. R. Acad. Sci. Paris **234**, 589–591 (1952).

Many, Abraham. An electrical network for determining the roots of a polynomial. Bull. Res. Council Israel **1**, 108–110 (1951).

Ashdown, G. L., and Selig, K. L. A general purpose differential analyser. I. Description of machine. The Elliott Journal **1**, 44–48 (1951).

Bruce, V. A dynamical analogue for investigating differential equations with periodic coefficients. Departments of Engineering and Mathematics, Stanford University, Stanford, Calif., 1948. 14 pp. (8 plates).

A pendulum of variable but controlled length is used as an analog to investigate solutions of the Mathieu equation. Experimental solutions are compared with those calculated from theory. Brief consideration is given to the effect of introducing certain types of non-linearity.
S. H. Caldwell (Cambridge, Mass.).

Swenson, George W., Jr., and Higgins, Thomas J. A direct-current network analyzer for solving wave-equation boundary-value problems. J. Appl. Phys. **23**, 126–131 (1952).

Seiwell, H. R. A new mechanical autocorrelator. Rev. Sci. Instruments **21**, 481–484 (1950).

Ball and disc mechanical integrators are used in a standard connection to obtain the integral of a product, where the terms of the product are the ordinates of two-identical plots of a time series. Computation of the autocorrelation function requires that this product integral be evaluated for a succession of time lags between the two plots. Details are given of the method used for hand computation of autocorrelation coefficients, the modifications of procedure required when the machine is used, and the method for calibrating the machine output. *S. H. Caldwell.*

✶Pleskot, Václav. Nomografie a grafický počet v technické praksi. [Nomography and graphical calculation in technical practice.] 2d ed. Knihovna Spolku Posluchačů a Absolventů Strojního a Elektrotechnického Inženýrství, spis čis. 103, Praha, 1949. 271 pp.

Although this planographed book does not offer new results, it gives a systematic treatment of the field which is rather complete. Historical notes and references to the literature are scattered throughout. After introducing scales and nets (20 pages) and treating intersection diagrams (54 pages with 17 completely executed charts) the various canonical forms for $F(x, y, z)=0$ are discussed in the chapter on alignment diagrams (106 pages with 25 completely executed charts) where attention is also given to duality, projective transformations, binary scales and nets, composite charts, and especially the problem of scale factors. This part closes with a short chapter on diagrams with sliding planes (six completely executed examples with detachable transparent planes) and a list of 75 classified problems. The last part (50 pages, practically independent of the first part) is devoted to graphical means for performing the operations of arithmetic and analysis, plotting of functions, and finding roots of equations. *R. Church* (Monterey, Calif.).

López Nieto, Antonio. Barycentric nomograms. Revista Mat. Hisp.-Amer. (4) **11**, 191–207 (1951). (Spanish)

In the plane five families of curves with parameters $z_1, \cdots, z_5$, the families (z_1, z_2), (z_3, z_4) forming two grids G_1, G_2 endowed with constant weights p_1, p_2, are named by the author a barycentric nomogram for $z_5=F(z_1, z_2, z_3, z_4)$, where F is an appropriate function. The value of z_5 is read at the center of masses p_1, p_2 placed respectively at (z_1, z_2), (z_3, z_4). The idea is extended by repetition to more than five variables and by allowing G_1 or G_2 to degenerate to four variables. The method is applied to an equation from engineering of the form $(1+f_1f_{23})z_5=g_1g_{23}$, where f_1, f_{23}, g_1, g_{23} are given functions of the variables indicated by the subscripts. *J. M. Thomas* (Durham, N. C.).

Jakobi, R. Zur Konstruktion der Achsen einer Ellipse. Z. Angew. Math. Mech. **32**, 30 (1952).

Reicheneder, Karl. Nadirketten mit Streckenmessung (Aeropolygonierung). Deutsche Akad. Wiss. Berlin. Veröff. Geodät. Inst. Potsdam, no. **6**, v+40 pp. (1951).

Zwinggi, Ernst. Ein weiteres Verfahren zur näherungsweisen Prämienbestimmung in der Invalidenversicherung bei Variation der Rechnungsgrundlagen. Mitt. Verein. Schweiz. Versich.-Math. **51**, 171–177 (1951).

The paper deals with the premium calculation (annual premiums) in permanent disability insurance when (1) it assumed that there are no recoveries, (2) the mortality is assumed to be the same for disabled as for active lives, (3) the probability of becoming disabled is given by Makeham's formula, (4) the number of all lives is (in the calculations) substituted for the number of active lives. An approximation is introduced with regard to the assumption (3). This approximation is shown to be closer than the one given in an earlier paper by the same author [same Mitt. **49**, 158–164 (1949); these Rev. **10**, 745].

K. Medin (Uppsala).

MECHANICS

***Corben, H. C., and Stehle, Philip. Classical Mechanics.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1950. xvii+388 pp. $6.50.

This book is an attempt to present classical mechanics in a way that smooths the transition to modern physics. This means that some topics are discussed with more than the usual emphasis. After introducing the fundamental notions and underlying assumptions of classical mechanics such as time, mass, momentum, force, energy, etc., Lagrange's equations for holonomic systems are derived. The chapter on the central force motion contains a discussion of scattering problems. The treatment of small oscillations of conservative systems, including molecular vibrations, and of the kinematics of rigid body motion is preceded by a chapter on linear vector spaces and matrices. Because of its vital importance in modern physics the various variational principles, Hamilton's equations, contact transformations and Poisson brackets are all discussed in detail. A brief discussion of the transition from discrete to continuous systems is given. Quasi-coordinates and quasi-momenta are introduced for the treatment of angular momentum. Noncentral forces are discussed in the case of the two-body problem. There is a brief introduction to relativistic mechanics with an application to the motions of ions in high energy accelerations. No quantum mechanics is discussed.

J. Haantjes (Leiden).

Gernet, M. M. Experimental determination of products of inertia, and dynamic balancing without a balancing machine. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **9**, no. 33, 39–52 (1950). (Russian)

The author proves that $I_{xy}=\frac{1}{2}(I_x+I_y)-I_{(xy)}$ where I_x, I_y and $I_{(xy)}$ are the moments of inertia about the x-axis, y-axis, and their bisectant. I_{xy} is the product of inertia Σmxy. This reduces experimental measurements to six (three about the axes, and three about their bisectants. He then turns to the problem of dynamic balancing of a uniformly rotating statically balanced rotor. The (rotating) reactions in the bearings A, B form a couple of moment $\omega^2\sqrt{(I^2_{yz}+I^2_{xz})}$, z being the axis of rotation. Since I_{yz}, I_{xz} can be determined by measurement, two positions of two balancing particles can be determined by computation alone. A discussion of the signs of their coordinates is necessary. A laboratory experiment is adduced with numerical data.

A. W. Wundheiler.

Mohr, Ernst. Die Bewegung eines Kreisels bei vorgegebenem Drehvektor. Math. Nachr. **6**, 1–10 (1951).

Bei der Integration der Differentialgleichungen von Frenet-Serret handelt es sich darum aus der Kenntnis des Darbouxschen Drehvektors $\mathfrak{d}=\tau\mathbf{t}+\kappa\mathbf{b}$ die Bewegung des Dreibeins $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ zu bestimmen. Es wird nun das Dreibein als Kreisel aufgefasst und versucht die Bewegung für den Fall zu bestimmen, wo der Drehvektor D eine beliebige Lage hat, sodass $\mathbf{D}=P\mathbf{t}+Q\mathbf{n}+R\mathbf{b}$ wo P, Q und R gegebene Funktionen der Zeit s sind. Verf. gibt vorerst eine einfache formale Lösung mit Reihen von mehrfachen Vektorprodukten und geht dann zu einer Darstellung mittels der Matrizenrechnung über.

O. Bottema (Delft).

García, Godofredo. The method of H. Wronski and the classical method of celestial mechanics. Actas Acad. Ci. Lima **13**, 45–68 (1950). (Spanish)

Villareal nell'anno 1899 e l'autore di questa nota negli anni 1922, 1924, 1931, 1933 si proposero di approfondire il principio teleologico che H. Wronski pose a fondamento di un suo metodo per lo studio dei problemi di dinamica celeste. Queste ricerche non raggiunsero pienamente lo scopo. In questo lavoro l'autore dice di essere riuscito ad interpretare il vero significato del principio di Wronski. Non sembra però che egli abbia esposto chiaramente la sua interpretazione.

Infatti, se si indica, come fa l'autore, con $M-S=\mathbf{Q}$ la coordinata vettoriale del pianeta M rispetto al Sole S, con $\mathbf{V}=d\mathbf{Q}/dt$, $d\mathbf{V}/dt=d^2\mathbf{Q}/dt^2$ la velocità e l'accelerazione assolute di M, con $\mathbf{Q}'$, $\mathbf{Q}''$ la velocità e l'accelerazione relative di M, rispetto ad un sistema di riferimento mobile avente l'origine in S, con $\boldsymbol{\psi}$ la velocità angolare di questo, è ben noto dagli elementi che la velocità assoluta $\mathbf{V}$ si ottiene con la formula $\mathbf{V}=\mathbf{Q}'+\boldsymbol{\psi}\wedge\mathbf{Q}$, e l'accelerazione assoluta si ottiene applicando lo stesso operatore a $\mathbf{V}$: $d^2\mathbf{Q}/dt^2=\mathbf{V}'+\boldsymbol{\psi}\wedge\mathbf{V}$. Non si comprende pertanto perché l'autore fa distinzione fra la velocità assoluta $\mathbf{V}$ e il vettore $\mathbf{Q}'+\boldsymbol{\psi}\wedge\mathbf{Q}$ che indica con $\boldsymbol{\omega}$ e chiama variabile canonica ausiliaria. Inoltre questo vettore ausiliario $\boldsymbol{\omega}$ viene applicato ad un punto N che l'autore non definisce (cfr. figura 1 di pag. 48). A pag. 46 nelle equazioni (5) e (6) sarebbe stato opportuno usare l'apice inveci di d/dt per distinguere la derivazione relativa dalla derivazione assoluta. A pag. 47 viene usato lo stesso versore $\mathbf{i}$ per due vettori $\mathbf{V}$ e $\mathbf{p}$ che hanno differenti direzioni.

G. Lampariello (Messina).

Mučnikov, V. M. On a general method of solution of the equation of motion of a train. Doklady Akad. Nauk SSSR (N.S.) **81**, 521–524 (1951). (Russian)

A train, considered as a uniform elastic cable with mass (locomotive) attached, passes with constant velocity over a track whose elevation is a continuous and piecewise linear function. Longitudinal displacements and stresses are studied. An interesting exercise in formal operational mathe-

matics, but several errors and omissions may lead to confusion. *R. E. Gaskell* (Bellevue, Wash.).

de Castro Brzezicki, A. Introduction to the dynamics of a point of variable mass. Revista Acad. Ci. Madrid **45**, 45–89 (1951). (Spanish)

The author considers a particle of variable mass m moving in accordance with one or the other of the vector differential equations of motion $d(mv)/dt=F$, $mdv/dt=F$. No general assumptions are stated concerning the variability of m, but the cases in which m is given as a function of v or of t are noted briefly. The chief features of the two theories of particle dynamics resulting from the above equations of motion are developed. In particular, the author discusses the Lagrangian and canonical forms of the equations of motion, variational principles, the Hamilton-Jacobi theory, and integrals of the equations of motion. Some of the results are illustrated by applications to a few of the classical problems. *L. A. MacColl* (New York, N. Y.).

Šul'gin, M. F. Generalization of Poisson's theorem to the case of holonomic non-conservative systems. Doklady Akad. Nauk SSSR (N.S.) **81**, 23–26 (1951). (Russian)

Let the Hamiltonian equations of a nonconservative holonomic dynamical system be

$$(1)\qquad \dot q_i=\partial H/\partial p_i,\quad \dot p_i=-\partial H/\partial q_i+Q_i,\quad i=1,\cdots,n,$$

where H and Q_i are known functions of the coordinates q_i, the impulses p_i and time t. Introduce supplementary coordinates u_i and impulses s_i and put

$$K=\sum_i(\partial H/\partial p_i)s_i+\sum_i(\partial H/\partial q_i-Q_i)u_i.$$

Then, with $K(t;q_i;p_i;u_i;s_i)$ as the Hamiltonian function, the system (1) assumes the form

$$(2)\qquad \dot q_i=\partial K/\partial s_i,\quad \dot p_i=-\partial K/\partial u_i,\quad i=1,\cdots,n,$$

and the u_i and s_i are determined by the equations

$$(3)\qquad \dot u_i=\partial K/\partial p_i,\quad \dot s_i=-\partial K/\partial q_i,\quad i=1,\cdots,n.$$

The results of the paper can be summarized as follows. Let $f(q_i;p_i;t)=\text{const.}$ and $\varphi(q_i;p_i;u_i;s_i;t)=\text{const.}$ denote two integrals of the system (1) and of the extended system (2) and (3) respectively. Then, if the substitution $u_i=\partial f/\partial p_i$, $s_i=-\partial f/\partial q_i$ is made, the equation $(f,\varphi)=\text{const.}$, where (f,φ) denotes the Poisson's bracket-expression of the functions f and φ, constitutes, in general, a new integral of the original system (1). If, in particular, $f(q_i;p_i;t)=\text{const.}$ is an integral of a nonconservative scleronomic system (1), then $\partial f/\partial t=\text{const.}$ is also an integral of this system, and consequently $\partial^2 f/\partial t^2=\text{const.}$, and so on, are integrals.

The theory is illustrated by an example of a free motion of a material point with unit mass in a resisting medium (the resistance being proportional to the velocity), and attracted by a force proportional to the distance from the origin. *E. Leimanis* (Vancouver, B. C.).

Hydrodynamics, Aerodynamics, Acoustics

Gerber, Robert. Sur l'existence des écoulements irrotationnels, plans, périodiques, d'un liquide pesant incompressible. C. R. Acad. Sci. Paris **233**, 1261–1263 (1951).

The author announces that, under certain restrictions, he is able to prove the existence of two-dimensional periodic gravity waves over a symmetric periodic bottom. The restrictions, aside from those mentioned in the title, are a limitation on the maximum slope of the bottom and on the value of a certain Froude number. The associated conformal mapping problem is reformulated as an integrodifferential equation which can be interpreted as an equation $x=F(x)$ in a Banach space where F is completely continuous. The restrictions mentioned allow application of the methods of Schauder and Leray. No details are given. *J. V. Wehausen* (Providence, R. I.).

Davies, C. N., and Aylward, Mary. The trajectories of heavy, solid particles in a two-dimensional jet of ideal fluid impinging normally upon a plate. Proc. Phys. Soc. Sect. B. **64**, 889–911 (1951).

In order to discuss theoretically the settling on a solid wall of fine particles in an incident air stream, the authors adopt the following procedure. They first determine by the classical Kirchhoff method the flow in a plane jet issuing from an orifice at a fixed distance from a plane wall and impinging normally on the wall. They then calculate by stepwise integration the motion of small particles in the flow, assuming that these do not affect the fluid motion and that the Stokes' formula governs the resistance of the particles in the fluid. The results of the calculations are discussed in detail with regard to dependence on the parameters. *D. Gilbarg* (Bloomington, Ind.).

Probstein, R., and Charyk, J. V. A method of solving the linear potential equation for axially symmetric flow. J. Aeronaut. Sci. **19**, 139–140 (1952).

James, D. G. Two-dimensional airfoils in shear flow. I. Quart. J. Mech. Appl. Math. **4**, 407–418 (1951).

The author presents a method for solving a two-dimensional steady flow past an airfoil in shear flow by means of Poisson integral. The flow field is first mapped into the region inside a unit circle and the airfoil goes over to the unit circle. For this region, a complex function whose imaginary part is the stream function can be uniquely defined in terms of the known boundary condition. In a special case of a circular cylinder, the solution obtained is compared with that of Tsien [Quart. Appl. Math. **1**, 130–148 (1943); these Rev. **5**, 21]. In addition, a general Joukowski airfoil is also considered and the effect of thickness and camber on forces and moment is examined. *Y. H. Kuo.*

Spreiter, John R., and Sacks, Alvin H. The rolling up of the trailing vortex sheet and its effect on the downwash behind wings. J. Aeronaut. Sci. **18**, 21–32, 72 (1951).

The rolling up of the trailing vortex sheet associated with wings of finite span is investigated theoretically and experimentally. First, similarity considerations give the relation

$$(*)\qquad \frac{e}{c}=K\frac{A}{C_L}\frac{b}{c},$$

where e is the distance for the completion of rolling up, c the chord, b the span, A the aspect ratio, C_L the lift coefficient, and K is a certain numerical constant. Next, the distance between the two rolled-up vortices, their downward velocity, and the radius of the vortex cores are obtained by impulse and energy considerations. The rolling up of vortex sheet is then considered by use of Betz's method [Z. Angew. Math. Mech. **12**, 164–174 (1932)], Westwater's numerical procedure [Air Ministry [London], Aeronaut. Res. Comm., Rep. and Memoranda, no. **1692**, 1936] and Kaden's analytical result [Ing.-Arch. **2**, 140–168 (1931)]. In particular,

the constant K in (*) is estimated as $K=0.28$ for the case of elliptic loading. Experimental studies are made by means of simple visual-flow experiments in a water tank using 8-inch span flat-plate airfoils of various plan forms, obtaining good agreement with theoretical predictions. Finally, formulas giving the downwash around a swept horseshoe vortex of arbitrary sweep are given for both subsonic and supersonic speeds.

I. Imai (Tokyo).

Sacks, Alvin H. Behavior of vortex system behind cruciform wings. Motions of fully rolled-up vortices. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2605**, 40 pp. (1952).

The motion of the four rolled-up vortices behind a low-aspect-ratio cruciform wing are studied both theoretically and experimentally. In the case of a triangular wing at 45° bank, the paths of the vortices have been traced and the distance at which the two upper vortices "leap-frog" through the lower two is calculated. It is found that this distance decreases with increasing lift coefficient C_L and increases with increasing aspect ratio A. For angles of attack up to a critical value of $C_L/\pi A^2$, the vortex motion is periodic in the downstream distance, while above the critical value it is aperiodic. Photographs showing the development of the wake at various stations behind the wing are presented. These were obtained by a flow-visualization technique utilizing the water surface.

Y. H. Kuo (Ithaca, N. Y.).

Muggia, Aldo. Sul calcolo dell'interferenza elica-ala. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **11**, 53–57 (1951).

To calculate the effects of a symmetrically located propeller slipstream at midspan on a wing of large aspect ratio, the author first assumes the slipstream jet to be cylindrical, of radius h, and to have a uniform airspeed SV, where V is the flying speed. The conditions of continuous pressure and flow direction at the edge of the jet, according to Koning [Aerodynamic theory, vol. 4, W. F. Durand, ed., Springer, Berlin, 1935] and others, yields conditions on the disturbance potential and normal derivative there. These are satisfied, within the accuracy of first-order wing theory, by a certain combination of trailing vortices and image trailing vortices, and also given by Koning [loc. cit.]. Thus, the downwash inside and outside the jet can be evaluated in terms of the derivative of the circulation distribution, $d\gamma/d\eta$, with attention to concentrated vortices trailing back from the points $\eta=\pm h$ where the wing cuts the jet boundary.

The Prandtl lifting-line problem, generalized to this case, is now attacked. The circulation distribution $\gamma(\eta)$ is divided into a continuous part γ_c and a discontinuous part γ_d. By the principle that the lift-loading itself must be continuous, the jump in γ_d at $\eta=\pm h$ is known in terms of $\gamma_c(\pm h)$, and a function having this jump is chosen. The continuous part γ_c produces a discontinuous downwash at $\eta=\pm h$ corresponding to the condition of discontinuous incidence existing there. Consequently, a technique used by Multhopp [Luftfahrtforschung **15**, 153–169 (1938)] in cases of discontinuous downwash is employed in evaluating γ_c. The results are not collected for convenient use, nor are any examples or numerical results provided. *W. R. Sears* (Ithaca, N. Y.).

Abdurahiman, P. V. Mathematical theory of cascade aerofoils. Math. Student **19**, 12–18 (1951).

Expository lecture.

Scholz, N. A method for calculating airfoils with prescribed pressure distribution. J. Aeronaut. Sci. **19**, 70–72 (1952).

Couchet, Gérard. Efforts aérodynamiques qui s'exercent sur le profil dans le cas de mouvements non stationnaires à circulation constante. C. R. Acad. Sci. Paris **234**, 808–810 (1952).

Further discussion of the author's earlier results [same C. R. **221**, 280–282 (1945); **223**, 974–976 (1946); O.N.E.R.A. Publ. no. 31 (1949); these Rev. **7**, 343; **8**, 295; **12**, 214].

E. Reissner (Cambridge, Mass.).

Moisil, Gr. C. An analogue to Galerkin's vector in the hydrodynamics of viscous liquids. Acad. Repub. Pop. Române. Bul. Şti. A. **1**, 803–812 (1949). (Romanian. Russian and French summaries)

For slow steady motions of viscous fluids the author shows that a formal solution analogous to Galerkin's for static elasticity may be obtained. He discusses the connection between this solution and the classical one in terms of biharmonic functions. *C. Truesdell* (Bloomington, Ind.).

Imai, Isao. On the asymptotic behaviour of viscous fluid flow at a great distance from a cylindrical body, with special reference to Filon's paradox. Proc. Roy. Soc. London. Ser. A. **208**, 487–516 (1951).

The method of Oseen for calculating the slow motion of a viscous incompressible fluid achieved great success in providing a description of the uniform flow past a cylinder apparently without divergences, and thereby overcoming the difficulties raised by the Stokes' paradox [see, e.g., Lamb, Hydrodynamics, 6th ed., Cambridge Univ. Press, 1932, pp. 614–617]. In order to calculate the force and moment on the immersed cylinder from momentum considerations, Filon [same Proc. **113**, 7–27 (1926); Philos. Trans. Roy. Soc. London. Ser. A. **227**, 93–135 (1928)] derived an asymptotic formula for the flow at large distances from the obstacle, basing his calculations on the first and second Oseen approximations. While these results gave a finite formula for the force on the body, the expression for the moment was infinite, thereby casting doubt on the suitability of the method of Oseen for treating the Navier-Stokes equations. In the present paper the author resolves "Filon's paradox" by showing that the next higher Oseen approximation suffices to determine a finite moment on the cylindrical obstacle. He finds by lengthy but skilful calculation the asymptotic formula for the stream function up to terms of order $O(r^{-1})$, this order of approximation being necessary for exact determination of the moment by the momentum flux argument. That the expression for the moment derived from this formula is now finite is seen to be a consequence of the fact that the term causing divergence of Filon's second approximation is precisely cancelled by a corresponding term in the third approximation.

D. Gilbarg (Bloomington, Ind.).

Sakadi, Zyurô. Motion of an incompressible viscous fluid between two concentric spheres. Math. Japonicae **2**, 71–74 (1951).

In the problem of the title one sphere is fixed and the other rotates with constant angular velocity. The author carries through the solution retaining terms of the second order in the velocities. He finds that the value of the moment on the fixed sphere is the same as in the linearized theory [cf. Lamb, Hydrodynamics, 6th ed., Cambridge,

1932, §334]. Retention of third order terms would have shown a change [cf. Toraldo di Francia, Boll. Un. Mat. Ital. (3) 5, 273–281 (1950); these Rev. 12, 871].

J. V. Wehausen (Providence, R. I.).

Taylor, Geoffrey. Analysis of the swimming of microscopic organisms. Proc. Roy. Soc. London. Ser. A. 209, 447–461 (1951).

The propulsion of a large object, such as a fish or ship, through a fluid is effected primarily through inertial reactions. As the author shows easily from the magnitudes involved, inertial stresses become trivially small compared to those due to viscosity in the case of self propelling microorganisms. In this paper the author proposes to account for the motion of these bodies, which are known to propel themselves by sending lateral waves down a thin tail, in terms of viscous stresses alone. He therefore idealizes the waving tail as a sheet that executes a plane sine wave motion in a viscous fluid, and assumes that the fluid motion thus induced is described by the biharmonic equation for the stream function, after neglect of the inertia terms in the Navier-Stokes equations. In approximate calculations the author shows that the sheet moves forward at a rate $2\pi^2b^2/\lambda^2$ times the velocity of propagation of the waves, where b is the amplitude and λ the wave-length. By considering the energy dissipated into the fluid and also the nature of the stresses in their dependence on the phase of two neighboring waving sheets, he finds that there is a strong reaction on the two surfaces to make the wave trains move in phase. This seems to account for the observed fact that the tails of neighboring spermatazoa pointed in the same direction wave in unison.

D. Gilbarg (Bloomington, Ind.).

Yamada, Hikoji. Theoretical estimation of meteorological high water. Rep. Res. Inst. Fluid Eng. Kyushu Univ. 6, no. 2, 22–33 (1950).

L'auteur étudie le comportement auprès des rivages des ondes soulévées dans un bassin (golfe au lac) par une bourrasque qui passe. D'après lui cette dénivellation provient de deux causes différentes: (1) soi-distant succion résultant de la perturbation de pression; (2) action directe du vent. L'auteur suppose qu'on peut négliger les termes non linéaires et traiter séparément les deux facteurs indiqués. En admettant le bassin rectangulaire à profondeur constante h, la vitesse V de bourrasque constante et parallèle à un de côtés de rectangle, on peut négliger la variation le long de l'autre côté de rectangle et représenter le processus par les équations

$$\frac{\partial u}{\partial t}=-g\frac{\partial \zeta}{\partial x}+\nu\frac{\partial^2 u}{\partial x^2}-g\frac{\partial P}{\partial x},\quad \frac{\partial \zeta}{\partial t}=-\frac{\partial}{\partial x}\int_0^h u\,dz,$$

avec les conditions frontières

$$-\frac{\nu}{gh}\left[\frac{\partial u}{\partial z}\right]_{z=0}=T,\quad [u]_{z=h}=0,\quad u=0 \text{ au rivage},$$

en désignant par X la coordonnée spatiale dans la direction du vent, z coordonnée verticale comptée en bas à partir du niveau moyen, u vitesse de l'eau dans la direction du vent, $\zeta(x, t)$ surface extérieur du liquide, ν viscosité cinématique que l'on suppose constante. Ces équations simplifiées étant encore trop difficiles, l'auteur introduit d'autres simplifications et arrive à une solution numérique.

V. A. Kostitzin (Paris).

Rotta, J. Statistische Theorie nichthomogener Turbulenz. I. Z. Physik 129, 547–572 (1951).

This paper deals with the statistical theory of inhomogeneous turbulence in shear flow with an approach and analysis similar in many ways to that used previously by P. Y. Chou [Quart. Appl. Math. 3, 38–54 (1945), compare eqs. (2.10), (2.17), (3.9) of the present paper with eqs. (1.6), (2.1), (5.1) in Chou's paper; these Rev. 6, 246]. Differential equations for the change of Reynolds stresses and kinetic energy are derived. The various terms occurring in these equations, corresponding to turbulent diffusion and other mechanisms, are then analyzed and simplified by physical reasoning and other approximations. In particular, the relation (4.10) is obtained with the help of modern concepts of turbulence and differs from eq. (6.14) of Chou's paper. After such simplifications, the theory is applied to calculate the statistical averages of various quantities for two-dimensional channel flow and compared with the experimental results of Reichardt and Laufer in a rectangular channel.

C. C. Lin (Cambridge, Mass.).

Chandrasekhar, S. The fluctuations of density in isotropic turbulence. Proc. Roy. Soc. London. Ser. A. 210, 18–25 (1951).

The author considers homogeneous isotropic turbulent motion of a compressible fluid. Let $\delta\rho$, $\delta\rho'$ be the deviations from the mean density at points P, P', distance r apart, and set $\tilde{\omega}(r, t)=\overline{\delta\rho\delta\rho'}$. Then the author finds an invariant, $\int_0^\infty r^2\tilde{\omega}(r, t)dr=$const., analogous to the Loĭtsyanskiĭ constant, and he interprets it physically. Under the assumption that the 4th order correlations can be expressed in terms of 2nd order correlations as with joint Gaussian distributions and that pressure and density are related adiabatically, he finds an equation of motion for $\tilde{\omega}(r, t)$. This equation is further simplified by assuming all velocities small compared to the velocity of sound c and using the defining scalars for $\overline{u_iu_j'}$ and $\overline{\rho u_i'}$ appropriate to an incompressible fluid. In this case it is shown that for large r density fluctuations are propagated with speed $v=\sqrt{2}[c^2+\frac{1}{3}\overline{u^2}]^{1/2}$.

J. V. Wehausen.

√★**Goldstein, S. Selected problems in gas dynamics.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2, pp. 280–291. Amer. Math. Soc., Providence, R. I., 1952.

In this paper, the author discusses various aspects of the theory of linearized supersonic flow of a compressible gas. The basic equation is the two-dimensional wave equation. After discussing the concepts of a thin body and a slender body, the boundary conditions for these bodies are considered. In particular, boundary conditions at leading and trailing edges, and supersonic and subsonic edges are discussed. Another section deals with a quadratic identity due to Ursell and Ward and its application to reversed flow theorems. Several methods of solution are briefly discussed: (1) the conical flows of Busemann and the generalizations of Lagerstrom; (2) the method of sources and sinks via Hadamard's solution; (3) the Heaviside operational method. Again, various improvements of the linearized theory are considered. First, a method where the potential is expanded in powers of the thickness and the logarithm of the thickness is mentioned. Secondly, a method used by Lighthill for improving the solution near a singular characteristic is noted. Finally, an example used by Whitham for studying the solution at infinity is considered.

N. Coburn.

García, Godofredo. The relation between pressure and density. Actas Acad. Ci. Lima **14**, 20–27 (1951). (Spanish)

Bishop, J. F. W. The application of virtual source distributions to problems in the linearized theory of supersonic flows. Quart. J. Math., Oxford Ser. (2) **2**, 291–307 (1951).

Perturbation velocity potential functions are usually obtained from source-sink distributions along the axis of a body of revolution, or in an equivalent layer approximately in the plane of a wing. Here the generalization to source-sink distributions on lines or planes which do not necessarily lie inside nor coincide approximately with the boundary has been called virtual. Several cases with undisturbed velocity parallel to the x-axis have been discussed. For example, a distribution on $x=0$ of constant density A (0) for $s \geqq (<)$ some constant a leads to a flow through a plane nozzle. Density $(s^2-a^2)^{\frac{1}{2}}$ for $-\infty \leqq s \leqq -a$ on $x=y=0$ yields a flow on a flat inclined plate which has a hyperbolic leading edge and is parallel to the y-axis. On $x=0$ the density As^2+By^2+C in the sector $|y/s| \leqq \sin \tau$, $s \leqq 0$, yields a flow at zero incidence over a symmetrical thick delta wing with supersonic leading edge. The supersonic trailing edge and the profile vary with A, B, C. Lift and drag coefficients have been computed as functions of thickness and incidence for the case of the straight trailing edge. The velocity components for the flat plate at incidence agree with results obtained by A. E. Puckett [J. Aeronaut. Sci. **13**, 475–484 (1946); these Rev. **8**, 109]. *J. H. Giese* (Havre de Grace, Md.).

Haack, Wolfgang. Charakteristikenverfahren zur näherungsweisen Berechnung der unsymmetrischen Überschallströmung um ringförmige Körper. Z. Angew. Math. Physik **2**, 357–375 (1951).

The velocity potential for linearized flow over a yawing pointed body of revolution can be computed from source-sink and dipole distributions on the axis of symmetry. For a body of revolution with a ring-shaped stabilizing surface or "shroud" this procedure has to be modified in the region influenced by the shroud. The author suggests continuing the flow calculation by a numerical method of characteristics. His computations of interference effects for a cylindrical shroud of zero thickness show that its lift is strongly influenced by the presence, shape, and location of the central body. *J. H. Giese* (Havre de Grace, Md.).

Ross, Frederick W. The propagation in a compressible fluid of finite oblique disturbances with energy exchange and change of state. J. Appl. Phys. **22**, 1414–1421 (1951).

Theory is presented for oblique shock waves involving (1) heat exchange, k_1, (2) transfer of part of the compressible fluid, k_2, to an incompressible state, and (3) change in specific heat ratios, k_3. By matching mass flow, momentum, and energy relations across an oblique finite disturbance it is shown that these additional conditions introduce new terms in the solution of the form $k_i u_1/(u_1-u_2)$. For k_i not zero a minimum shock angle is found which always exceeds the Mach angle by a definite increment. (From the author's summary.) *Y. H. Kuo* (Ithaca, N. Y.).

Shen, S. F., and Lin, C. C. On the attached curved shock in front of a sharp-nosed axially symmetrical body placed in a uniform stream. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2505**, 66 pp. (1951).

The authors study the curvature of an attached shock at the vertex of a body of revolution. Their method employs a perturbation scheme based on the Taylor-Maccoll conical flow and essentially presupposes an asymptotic form of the flow near the vertex. The first order approximation yields the value of the ratio of the radius of curvature of the body to that of the shock wave at the vertex; this is computed for a range of Mach numbers for each of three vertex angles and is shown to reproduce qualitatively the corresponding results for two dimensions obtained by Thomas [J. Math. Phys. **27**, 279–297 (1949); these Rev. **10**, 494]. The authors note that the ratio vanishes as the free stream Mach number approaches a certain value for which the flow behind the shock is subsonic. The same result has been obtained with different methods by Cabannes [C. R. Acad. Sci. Paris **232**, 481–483 (1951); these Rev. **12**, 767]. The corresponding situation occurs also in two dimensions, as first observed by Crocco, and has recently been fully explored by Cabannes [Publ. Sci. Tech. Ministère de l'Air, Paris, no. 250, 181–196 (1951); these Rev. **13**, 399]. There are two anomalous results, namely, the divergence of the calculations for bodies convex at the vertex and the required presence of singularities in the shape of either the body or shock wave as a consequence of the higher order approximations. As the authors observe, these results cast some doubt on the validity of their methods. *D. Gilbarg.*

Cabannes, Henri. Étude de l'onde de choc attachée dans les écoulements de révolution. I. Cas d'un obstacle terminé par une ogive. Recherche Aéronautique **1951**, no. **24**, 17–23 (1951).

For a pointed body of revolution in symmetric flow, a calculation of conditions close to the nose is undertaken by means of series expansion. The values of velocity components, density, etc. at the nose are those for flow past a right circular cone, which have been calculated and tabulated by several authors; linear equations for the next terms are obtained. In particular, the ratio of the meridian curvatures of the body and the attached shock wave at the nose is evaluated. This ratio is tabulated and plotted for the case of a 20° nose semi-angle and is compared with the analogous plane-flow case. The appearance of negative values of the ratio at relatively low Mach numbers is discussed briefly, and will be considered more fully in a later paper. The case of a body whose meridian curvature is zero at the nose is treated separately.

In a note added after the paper was edited, some additional values of the ratio of meridian curvatures are given, supplied by C. C. Lin. These are for nose semi-angles of 10°, 20°, and 30°, and for a range of Mach numbers. These data and those presented earlier for 20° are in rough agreement. *W. R. Sears* (Ithaca, N. Y.).

Fletcher, C. H., Taub, A. H., and Bleakney, Walker. The Mach reflection of shock waves at nearly glancing incidence. Rev. Modern Physics **23**, 271–286 (1951).

Survey article. *M. J. Lighthill* (Manchester).

Ludford, G. S. S. The boundary layer nature of shock transition in a real fluid. Quart. Appl. Math. **10**, 1–16 (1952).

La théorie de l'onde de choc dans un milieu compressible est faite en partant des équations de Navier-Stokes qui gouvernent le mouvement des fluides réels. Les méthodes utilisées sont apparentées à celles de la théorie de la couche limite de Prandtl et les résultats suivants sont établis: étant donnée une surface $S(x, y, z, t)$ dépendant du temps t, les équations de Navier admettent une classe de solutions qui

ont les propriétés suivantes. 1) Si le coefficient de viscosité μ est petit ces solutions ont sur S des dérivées normales qui sont grandes et des dérivées tangentielles qui sont petites. 2) Lorsque μ tend vers zéro S devient une surface de discontinuités sur laquelle les conditions de Rankine-Hugoniot sont vérifiées. L'onde de choc apparait ainsi comme un phénomène asymptotique des fluides réels lorsque la viscosité devient nulle. *R. Gerber* (Grenoble).

Lin, C. C., and Shen, S. F. Studies of von Kármán's similarity theory and its extension to compressible flows. I. A critical examination of similarity theory for incompressible flows. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2541**, 24 pp. (1951).

Lin, C. C., and Shen, S. F. Studies of von Kármán's similarity theory and its extension to compressible flows. II. A similarity theory for turbulent boundary layer over a flat plate in compressible flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2542**, 38 pp. (1951).

Shen, S. F. Studies of von Kármán's similarity theory and its extension to compressible flows. III. Investigation of turbulent boundary layer over a flat plate in compressible flow by the similarity theory. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2543**, 43 pp. (1951).

The purpose of the present series of reports is to extend von Kármán's similarity hypothesis of turbulent shear flow [Nachr. Ges. Wiss. Göttingen **1930**, 58–76] to the compressible turbulent boundary layer over a flat plate. To provide for a sound foundation for the intended development, the authors first examine critically the similarity hypothesis from the point of view of modern statistical theory of turbulence [cf. von Kármán and Lin, Advances in Applied Mechanics, vol. 2, pp. 1–19, Academic Press, New York, 1951; these Rev. **13**, 83]. The important result here is the conclusion that the neglect of viscous stresses in the momentum equations means essentially the emphasis on large scale eddies, and therefore in formulating the similarity concept only the lowest order deviatives of velocity profile should be used. As a consequence of this, the so-called τ-theory was shown to be the correct one to use in connection with von Kármán's similarity hypothesis.

In part II, the authors first show the importance of viscous dissipation in the energy equation for boundary layer flow. In fact, for an insulated wall, the magnitude of the dissipation terms is estimated to be of the same order as that of the convection terms. The inclusion of viscous dissipation in the energy equation destroys the possibility of a "isoenergy relation" so frequently used by the previous investigators of this problem. To carry out the analysis based upon the similarity concept, the authors have to introduce another assumption: The percentage fluctuations in temperature, density and pressure are small. The equations are partially linearized using this assumption. It is thus clear that the present theory of the turbulent boundary layer is not applicable to flows at high supersonic Mach numbers, or to hypersonic flows. The analysis gives two length-scales, one from the velocity profile, one from the temperature profile. The choice can be decided by the emphasis on either the momentum equation, or the energy equation.

In part III, the theory developed in part II is applied to the boundary layer over a flat plate. Here the calculation is simplified on the explicit assumption that the average quantities are functions of only y, the distance from the wall. A temperature-velocity relation is first deduced using the energy equation,

$$\frac{d\theta}{d\bar{u}}=M_1^2(A_1\bar{u}+A_2\sqrt{\theta})+Q$$

where θ, $\bar{u}$ and Q are non-dimensionalized temperature, velocity and heat flux of the boundary layer. A_1 and A_2 are constants, and M_1 is the free stream Mach number. If A_1 is zero or negligibly small, the relation is very similar to the "isoenergy relation". However, the size of A_2 is not immediately determined by the theory. The velocity and the temperature profiles are then calculated for two cases: a) subsonic flow, and b) general case. For a), the computation is done by expansion in power series of M_1^2. However, for lack of accurate experimental results to fix the constants necessary for the theory, no numerical results useful for engineers can be given.

These reports seem to contain a number of misprints.
H. S. Tsien (Pasadena, Calif.).

Fenain, M. Trainée d'onde d'ailes en flèche effilées à profils évolutifs. Recherche Aéronautique **1951**, no. 24, 25–37 (1951).

In an earlier paper [same journal **1950**, no. 16, 27–38; these Rev. **12**, 139] the same author applied the theory of generalized conical (linearized) flows developed by Germain [ibid. **1949**, no. 7, 3–16; these Rev. **10**, 492] to calculate the wave drag at zero incidence of a class of delta wings. Here he extends this work to tapered sweptback wings. Truncated tips are produced by a method of superposition, as are subsonic trailing edges. The particular family treated, after more general introductory paragraphs, are wings whose surfaces are formed by four pieces of "hyperbolic paraboloids":

$$x_3=\text{const.}\times(x_1-x_2\cot\gamma)(x_2\cot\omega+c-x_1)$$

where x_1, x_2, x_3 are rectangular Cartesian coordinates, x_1 lying in the direction of the stream and x_2 spanwise, γ and ω being the complements of the angles of sweepback of the leading and trailing edges, and c the root chord.

The surface pressure coefficient of these wings is worked out. The drag has been calculated and is presented in a series of graphs. The graphs cover the range of taper ratios (ratio of tip chord to root chord) from 0 to 0.8, and a range of the ratio $\tan\gamma/\tan\omega$ from 0 to 0.8. The untapered wing, for which both ratios are equal to 1, yields formulas in agreement with Harmon's [Tech. Notes Nat. Advis. Comm. Aeronaut., no. **1449** (1947)], and his numerical calculations have not been duplicated. The abscissa of the charts is $(M^2-1)^{1/2}\tan\gamma$, M being the stream Mach number. For one case, isobars of the wing surface are shown. Finally, the results are discussed from the practical engineering standpoint. *W. R. Sears* (Ithaca, N. Y.).

Hsu, E. H. A general equation of horizontal mass divergence in the atmosphere. J. Meteorol. **8**, 395–397 (1951).

A general formula for the horizontal mass divergence of an atmospheric motion is obtained without approximation from the equations of motion. The effect of friction is neglected but the curvature of the Earth's surface is taken into account. The horizontal mass divergence is expressed as the sum of fourteen terms, eleven of which are new. The effects of horizontal motion and of vertical motion are included in this expression. Previously obtained formulae, due to other authors, for the cases of gradient and of geostrophic

flow are obtained as special examples, and the circular symmetric vortex is also dealt with. It is estimated that nine of the fourteen terms found are significant in naturally occurring motions. *G. C. McVittie* (London).

Hollmann, Günther. Beitrag zu einer Theorie der Entstehung ortsfester Druckgebilde. I. Z. Meteorologie **5**, 258–267 (1951).

A frictionless baroclinic quasi-geostrophic 3-dimensional model of long waves (upwards of 2500 km) in the westerlies is constructed and its dynamic stability is investigated. No stratosphere is assumed in the boundary conditions. It is shown that such planetary waves can be unstable, provided the meridional temperature gradient takes on values in a definite interval determined by the wavelength. Pressure patterns of large area build up more rapidly than do smaller ones, but require a stronger meridional temperature gradient. The buildup of the wave pattern is accomplished in the time interval of a single day. In this process of amplitude increase, a coupling or feedback mechanism between upper and lower levels is shown to be effective. It is further shown that purely dynamic considerations are not sufficient to determine stability; the location of heat sources must also be considered. Finally, it is stated (without proof) that the stratosphere, when added to the model, exerts a damping effect on wave development. No reference whatsoever is made to the works of Charney [J. Meteorol. **4**, 135–162 (1947); these Rev. **9**, 163] and Eady [Tellus **1**, no. 3, 33–52 (1949); these Rev. **13**, 86], to which the results of this paper are closely related. *W. D. Duthie* (Monterey, Calif.).

Brillouin, Jacques. Rayonnement transitoire des sources sonores et problèmes connexes. Ann. Télécommun. **5**, 160–172, 179–194 (1950).

The author starts by remarking that the field, created by an oscillating source has 2 parts: the first part is constituted by progressive waves which radiate into space; the second part is stationary and does not entail transport of energy. The question is, how this second part of the field comes into being, when the source starts and how it is extinguished when the source stops. This problem is treated in simple cases: an oscillating sphere and a pulsating sphere. The method used is the consideration of the eigen-oscillations corresponding to the space outside such a sphere. These eigen-oscillations are severely damped. The symbolic calculus is also used. The further treatment may be said to be classical and results are shown numerically as well as in curves. The author then treats a rigid sphere submitted to an external force. This is a diffraction problem which has been solved in several older papers and it is treated briefly by the author. Another diffraction problem treated is a row of obstacles of regular spacing on which a plain wave strikes. The diffraction of a plain wave by a microphone is also treated. *M. J. O. Strutt* (Zurich).

Brehovskih, L. M. The diffraction of sound waves from an uneven surface. Doklady Akad. Nauk SSSR (N.S.) **79**, 585–588 (1951). (Russian)

The problem is that of a plane wave incident upon a doubly-corrugated surface, given in rectangular coordinates by $X_3=\zeta(X_1, X_2)$ where ζ is periodic in both arguments. In the formula

$$\varphi(P)=\frac{1}{4\pi}\int_S\int\left[\varphi\frac{\partial}{\partial n}\left(\frac{e^{ikR}}{R}\right)-\frac{e^{ikR}}{R}\frac{\partial\varphi}{\partial n}\right]dS$$

he substitutes for φ the sum of the incident and reflected waves with a general reflection coefficient; the limits of applicability of this procedure are specifically excluded from the investigation. For $\exp(jkR)/R$ he substitutes a double integral in terms of plane waves. After some formal manipulations there results an expression for φ as a double series of plane waves. A procedure is given for determining the amplitudes of these waves. Special cases discussed include the case of a small ripple upon a large one. *F. V. Atkinson* (Ibadan).

Mičurin, V. K., and Černov, L. A. The scattering of sound in dispersed systems. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 920–926 (1951). (Russian)

Si suppone che in un mezzo omogeneo siano sospese particelle sferiche di una sostanza diversa. Viene esaminato il caso in cui le particelle hanno densità molto maggiore del mezzo e sono incompressibili, come pure il caso delle emulsioni, in cui le densità sono quasi eguali, ma la compressibilità delle particelle può essere molto superiore a quella del mezzo. Gli autori dimostrano che nel primo caso è necessario tener conto del tempo di rilassamento, relativo alle oscillazioni forzate delle particelle nel mezzo, altrimenti si ottengono risultati errati per il coefficiente di diffusione del suono. La cosa viene esemplificata numericamente nel caso della nebbia. *G. Toraldo di Francia* (Firenze).

Levin, M. L. On the scattering of sound in a weakly nonhomogeneous medium. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 937–939 (1951). (Russian)

Viene applicato il ben noto metodo di perturbazione di prima approssimazione (quella che nella meccanica quantistica è l'approssimazione di Born). *G. Toraldo di Francia* (Firenze).

Rževkin, S. N. The connection of the problem of sound diffraction by a sphere with the reciprocity theorem. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 1224–1227 (1951). (Russian)

The point A is on the surface of a fixed rigid sphere, the point B at a large distance from the sphere. The author verifies a reciprocity result for the pressure produced at A or B by point sources at B or A; this he does by direct solution of the boundary problems, subject to approximations on account of the large distance. *F. V. Atkinson.*

Markham, Jordan J., Beyer, Robert T., and Lindsay, R. B. Absorption of sound in fluids. Rev. Modern Physics **23**, 353–411 (1951).

Expository paper.

Poincelot, Paul. Sur la notion de vitesse de groupe. C. R. Acad. Sci. Paris **234**, 599–602 (1952).

Elasticity, Plasticity

√★**Timoshenko, S., and Goodier, J. N. Theory of Elasticity.** 2d ed. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1951. xviii+506 pp. $9.50.

The new edition of this well known book contains several important additions, as stated in the preface:

"The treatments of the photoelastic method, two-dimensional problems in curvilinear coordinates, and thermal stress have been rewritten and enlarged into separate new

chapters which present many methods and solutions not given in the former edition. An appendix on the method of finite differences and its applications, including the relaxation method, has been added. New articles and paragraphs incorporated in the other chapters deal with the theory of the strain gauge rosette, gravity stresses, Saint-Venant's principle, the components of rotation, the reciprocal theorem, general solutions, the approximate character of plane stress solutions, center of twist and center of shear, torsional stress concentration at fillets, the approximate treatment of slender (e.g., solid airfoil) sections in torsion and bending, and the circular cylinder with a band of pressure.

"Problems for the student have been added covering the text as far as the end of the chapter on torsion."

The engineer will find here the necessary fundamental knowledge of the theory of elasticity together with so many practical applications that the book will provide for him an excellent compendium making the design of most engineering structures much easier. The interesting pedagogic problems will increase the value of this book for students of engineering. Clearly, within the frame of this work, authors have not been able to include the most recent developments of the theory. Neither is the mathematical rigor always impeccable, but that should not seriously reduce the value of this book. *F. Niordson* (Stockholm).

***Murnaghan, Francis D. Finite deformation of an elastic solid.** John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. v+140 pp. $4.00.

After the inevitable introductory chapter containing pure mathematics neither necessary nor sufficient for the remainder of the book, the author develops the classical theory of finite strain. Results not already published elsewhere are: determination of the special forms assumed by the cubic terms in the strain energy for aelotropic materials; second order effects in shearing stress (not simple shear); second order change in length and diameter of a circular cylinder in torsion. For some of the other special cases in which second order effects are discussed fully general solutions occur in the literature. The author gives no references except to his own text books, and the presentation in matrix form is not only unnecessarily elaborate but also far from giving a complete idea of the present state of the theory of finite elastic strain. *C. Truesdell* (Bloomington, Ind.).

Tolotti, Carlo. Problemi aperti della teoria delle deformazioni elastiche finite. Atti del Terzo Congresso dell' Unione Matematica Italiana, Pisa, 1948, pp. 52–62. Casa Editrice Perrella, Rome, 1951. 1600 Lire.

Expository lecture.

Kondo, Kazuo. The geometry of the perfect tension field. III. J. Jap. Soc. Appl. Mech. 4, 101–104 (1951). (English. Japanese summary)

The author continues his researches on stress fields in which two principal stresses vanish at each point [same J. 3, 36–39, 85–88, 96 (1950); these Rev. 12, 63, 301]. He now considers some of the ruled surfaces tangent to such fields. *C. Truesdell* (Bloomington, Ind.).

Moisil, Gr. C. Two applications of areolar polynomials of order 1. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 125–128 (1950). (Romanian. Russian and French summaries)

An areolar polynomial of first order is a function $f(z)$, $z=x+iy$, which satisfies the equation $(\partial/\partial x+i\partial/\partial y)^2 f=0$. The Airy function Ω is assumed to give the distribution of stresses in a 2-dimensional deformable medium. If Ψ is the stream function for the plane flow of an incompressible viscous fluid, in steady slow motion, then $\Omega+2\mu i\Psi$ (μ = coefficient of viscosity) is an areolar polynomial of order 1. If φ and ψ are the functions defined by

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(\varphi-i\psi)=u+iv,$$

where (u, v) are the displacements in an elastic body in plane strain, then $H=\Omega+2(\lambda+\mu)\varphi$ is harmonic. Denoting by K the conjugate of H and constructing $\Psi=\psi+\frac{1}{2}(\lambda+\mu)^{-1}K$, it is shown also that $\Omega+2\mu(\lambda+\mu)(\lambda+2\mu)^{-1}i\Psi$ is an areolar polynomial of order 1. *G. Béguin* (Ann Arbor, Mich.).

Moisil, Ana. On a vector analogous to Galerkin's vector in the equilibrium of elastic bodies with transverse isotropy. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 207–210 (1950). (Romanian. Russian and French summaries)

Galerkin's solution [C. R. Acad. Sci. Paris 190, 1047–1048 (1930)] of the equilibrium equations of elasticity is generalized to the case of a body with only one axis of elastic symmetry. *C. Truesdell* (Bloomington, Ind.).

Ionescu-Cazimir, Viorica. On the equations of thermoelastic equilibrium. I. The analogue of Galerkin's vector. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 589–595 (1950). (Romanian. Russian and French summaries)

The author claims to obtain a general solution of Galerkin type for bodies subject to infinitesimal thermoelastic stress. The reviewer notes that the author's equations (1) are suitable only for equilibrium, while equation (4) and the remainder of the analysis refers to motion. *C. Truesdell.*

Iacovache, Maria. On an extension of Galerkin's method to the system of equations of elasticity. Acad. Repub. Pop. Române. Bul. Şti. A. 1, 593–596 (1949). (Romanian. Russian and French summaries)

The author shows that Galerkin's method of solution [C. R. Acad. Sci. Paris 190, 1047–1048 (1930)] for the statical equations of infinitesimal elasticity theory can be extended to the dynamical case in a medium of Galitzin's type [Bull. Acad. Imp. Sci. St.-Pétersbourg (6) 1912, 219–236], where a damping force per unit volume of magnitude $\nu\partial u/\partial t$ is assumed. The author's result is

$$u=\left[(\lambda+2\mu)\nabla^2-\rho\frac{\partial^2}{\partial t^2}-\nu\frac{\partial}{\partial t}\right]\Phi-(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\Phi,$$

where the Galerkin vector Φ satisfies

$$\left[(\lambda+2\mu)\nabla^2-\rho\frac{\partial^2}{\partial t^2}-\nu\frac{\partial}{\partial t}\right]\left[\mu\nabla^2-\rho\frac{\partial^2}{\partial t^2}-\nu\frac{\partial}{\partial t}\right]\Phi=0.$$

C. Truesdell (Bloomington, Ind.).

Iacovache, Maria. On small motions of an elastic body in the case of a distribution of spherical or cylindrical tensions. Acad. Repub. Pop. Române. Bul. Şti. Ser. Mat. Fiz. Chim. 2, 597–601 (1950). (Romanian. Russian and French summaries)

For the small motions of an elastic body, the author claims to determine the most general state of pure tension in two cases: $\sigma_x=\sigma_y=\sigma_z$ and $\sigma_x=\sigma_y$. The reviewer notes that the results in the latter case do not reduce to those of the former when one sets $\sigma_z=\sigma_x$. *C. Truesdell* (Bloomington, Ind.).

Melan, Ernst. Temperaturverteilungen ohne Wärmespannungen. Österreich. Ing.-Arch. **6**, 1–3 (1951).

An elastic body of homogeneous isotropic material is in equilibrium under zero body force and zero surface tractions; the strain due to the steady temperature field T (free from singularities) is small. If the stress field σ_{ij} is zero, then it may be proved that $T=a+b_ix_i$, where a is a constant scalar, b_i is a constant vector, and the x_i's are rectangular cartesian coordinates; the formula for T given by the author is incorrect, and the error lies in the neglect of certain strain compatibility equations. Suppose that the body is a plane plate bounded by parts of the surfaces with equations $x_3=h$, $x_3=-h$ and $f(x_1, x_2)=0$. From the author's analysis, it appears probable to the reviewer that, if $(\partial^2/\partial x_1^2+\partial^2/\partial x_2^2)T(x_1, x_2)=0$, then, except perhaps near the plate edges, the σ_{ij}'s may be made arbitrarily small by making h small enough, f and T remaining unaltered. The author, however, deduces from his analysis that a plane harmonic temperature field and a zero stress field can coexist in a thin plate. In general, this result contradicts the previous result; it is only an approximation (although probably a very good one if the plate is sufficiently thin) involving errors similar to those in generalized plane stress.

H. G. Hopkins (Manchester).

Platone, Maria Giovanna. Sugli stati di tensione piana in un corpo cilindrico elastico. Ann. Scuola Norm. Super. Pisa (3) **5**, 57–70 (1951).

A. Ghizzetti [Ann. Mat. Pura Appl. (4) **29**, 125–130 (1949); these Rev. **12**, 63] introduced a special class of plane stress problems and gave a general solution in terms of arbitrary harmonic functions. The author determines these functions in such a way that the normal component of stress on a circular cylindrical boundary assumes prescribed values.

C. Truesdell (Bloomington, Ind.).

Parkus, H. Die Grundgleichungen der allgemeinen Zylinderschale. Österreich. Ing.-Arch. **6**, 30–35 (1951).

The author's previous formulation of the theory of thin elastic shells [same journal **4**, 160–174 (1950); these Rev. **11**, 701] is specialized to the case when the middle section is a cylinder.

C. Truesdell (Bloomington, Ind.).

Yen, Kuo Tai, Gunturkun, Sadettin, and Pohle, Frederick V. Deflections of a simply supported rectangular sandwich plate subjected to transverse loads. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2581**, 39 pp. (1951).

Differential equations that take into account the effect of shear deformation in the core are solved to determine the deflection of a rectangular plate of sandwich construction subjected to a distributed or a concentrated load and having simply supported edges. Curves are presented showing the results for square sandwich plates.

H. W. March.

Paschoud, J. Le calcul numérique de la flexion circulaire des plaques de révolution d'épaisseur variable. Schweiz. Arch. Angew. Wiss. Tech. **17**, 305–312 (1951).

L'étude de la fléxion de ces plaques est ramenie à un système de deux equations intégrales liant le moment fléchissant radial et la rotation de la fibre radiale moyenne. Ce système est résolu par itération. Deux constantes auxiliaires doivent être déterminées à chaque pas pour que les inconnues restent finies au centre de la plaque. Les résultats concordent avec ceux donnés par Favre pour le même problème [Bull. Tech. Suisse Romande **75**, 225–230 (1949)].

J. Kuntzmann (Grenoble).

Ashwell, D. G. The axis of distortion of a twisted elastic prism. Philos. Mag. (7) **42**, 820–832 (1951).

A theory of the twisting of thin-walled, prismatic, elastic solids is given under the following assumptions: (1) The stresses in longitudinal filaments = their strain × Young's modulus. (2) The strains e_0 and $(r\tau)^2$ are small fractions (e_0 the strain of the filament lying along the axis, τ the twist per unit length, r the distance of elements of cross-section from its centroid). The author shows that both conditions are satisfied for any thin-walled prism, except if its material is distributed at a fairly constant distance from the centroidal axis. The main result of the theory is that the axis of distortion of such a prism is, in general, quite different from the "axis of displacement", an effect predicted by M. S. G. Cullimore [Engineering structures, Academic Press, New York, 1949, pp. 153–164]. The authors made experiments with a symmetrical angle-shaped and a symmetrical U-shaped (channel) prism both of beryllium-copper, and a steel pocket tape measure having a transverse curvature for stiffness. The experimental result fully confirmed the author's theory.

P. Neményi (Washington, D. C.).

Lattanzi, Filippo. Applicazione della teoria dell'ellisse di elasticità trasversale allo studio di un'asta curva elasticamente vincolata agli estremi. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 395–400 (1951).

On considère une barre courbe dont une extrémité est encastrée; on suppose qu'à l'égard des extrémités de la barre on connalt l'ellipse d'élasticité transversale définie en 1911 par C. I. Ricci; on suppose alors qu'un système de forces parallèles est donné normale au plan de la barre; on cherche une expression pour la rotation angulaire à l'extrémité fixe et pour le déplacement de l'extrémité libre.

B. Levi.

Benscoter, Stanley Urner. Secondary stresses in thin-walled beams with closed cross sections. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2529**, 104 pp. (1951).

An accurate method of determining secondary stresses in thin-walled, uniform beams of closed cross-section is presented. The cross-sections are assumed to be preserved by closed spaced rigid diaphragms. In the first section of the paper the integro-differential equation governing axial displacements is formulated and solved for a beam without longitudinal stiffeners. In the second section of the paper the corresponding summation-difference equation is developed and solved for a beam with stiffeners (flanges and stringers). The cross-section, loading distribution, and end conditions are assumed to be arbitrary. (Author's summary.)

H. G. Hopkins (Manchester).

Nardini, Renato. Sull'equazione del moto di una trave elastica con ereditarietà. Atti Sem. Mat. Fis. Univ. Modena **4**, 68–87 (1950).

A partial integro-differential equation for the motion of beams according to V. Volterra's accumulative theory of elasticity was derived by E. Volterra [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 42–47, 178–180 (1947); these Rev. **8**, 546]. D. Graffi [same Atti **3**, 227–247 (1949); these Rev. **11**, 483] asserted without proof the existence and uniqueness of a solution corresponding to a deflection prescribed up to a certain time, the two ends of the beam being pinned. The author proves that a more general class of equations possesses a unique suitably smooth solution corresponding to prescribed initial deflection and velocity for the entire beam, the end points again being

pinned. He also formulates conditions sufficient that a series solution given by Graffi for the case when the nucleus of accumulation is a finite sum of exponentials be absolutely convergent. *C. Truesdell* (Bloomington, Ind.).

Hidaka, Koji. Vibration of a square plate clamped at four edges. Math. Japonicae **2**, 97–101 (1951).

The aim of this paper is to calculate the approximate value of λ for the fundamental mode belonging to the boundary-value problem in which W satisfies the differential equation $\nabla^4 W - \lambda W = 0$ inside a unit square and the boundary conditions $W = \partial W/\partial n = 0$ on the edges of the square. He replaces the differential equation by a difference equation containing values of W at the nodal points of square grid into which the original square is divided, and determines the lowest characteristic number for this difference equation. This is done for five different sizes of mesh, and then in effect the supposed true value is obtained by extrapolation from the five thus computed. The result secured is $\lambda = 13.2992\pi^4$ with an error which the author believes to be about 0.033%. *W. E. Milne* (Corvallis, Ore.).

Holden, A. N. Longitudinal modes of elastic waves in isotropic cylinders and slabs. Bell System Tech. J. **30**, 956–969 (1951).

Longitudinal elastic waves in rods and slabs are considered in which the motion of a cross-section normal to the propagation direction contains several nodes. The mode without nodes is the longitudinal wave as normally considered, and a single node gives flexural waves. Higher order modes symmetrical about the centre are considered. Full treatment of possible modes and frequencies for the slab is first given to supply an analysis simpler but analogous to the cylindrical rod which is treated briefly by comparison. *E. H. Lee.*

Ghosh, M., and Ghosh, S. K. Dynamics of the vibration of a bar excited by the longitudinal impact of an elastic load. Indian J. Phys. **25**, 153–162 (1951).

The impact of a mass cushioned by a massless spring striking longitudinally the end of an elastic rod is considered. Two cases are treated, the other end of the rod being fixed and free. An operational type of solution is applied with expansion in terms of negative exponentials to give the successively reflected component waves. The form of the results is discussed briefly. *E. H. Lee.*

Gassmann, Fritz. Über Dämpfung durch Abstrahlung elastischer Wellen und über gedämpfte Schwingungen von Stäben. Z. Angew. Math. Physik **2**, 336–356 (1951).

A vibrator in the form of an elastic rod is considered attached to an elastic support. Motion of the interface, which is assumed to be rigid, is considered to link the two parts of of the system. Steady sinusoidal motion is treated. The characteristics of the support are represented by dynamic spring constants for a general force and moment acting at the interface. Special cases are discussed of a semi-infinite rod and an infinite elastic medium for the support. The vibrator in the form of a rod is analysed including both external and internal damping. The latter is expressed by including stress and strain rate terms in the stress-strain relation. Longitudinal, bending and torsional deformations are considered. In addition to internal and external damping of the vibrator, radiation damping occurs due to energy propagated into the support. For small damping, evaluation of damping coefficients is discussed from the study of resonance curves. *E. H. Lee* (Providence, R. I.).

✱Saibel, E. Free and forced vibrations of composite systems. Proceedings of the Symposium on Spectral Theory and Differential Problems, pp. 333–343. Oklahoma Agricultural and Mechanical College, Stillwater, Okla., 1951. $3.00.

This is mainly an expository paper in which the eigenvalue theory of linear, ordinary and partial, differential equations is applied to the following engineering problems: (a) Determination of natural frequencies ω^2 of the vibration of a shaft carrying a number of masses. Here a method is described in which a trial value of ω^2 is selected and in a series of linked calculations a 'residual' is carried along the shaft from mass to mass with the aim of so selecting the trial ω^2 that the final residual is 0. (b) Determination of the natural frequencies of the vibration of a continuous beam. This follows the standard method of separation of variables, and the solution of a number of linear boundary value problems by expansion into eigenfunction series is discussed. (c) Finally problems for the forced vibrations of a continuous beam are sketched. *H. O. Hartley* (London).

Johnson, Aldie E., Jr., and Buchert, Kenneth P. Critical combinations of bending, shear, and transverse compressive stresses for buckling of infinitely long flat plates. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2536**, 40 pp. (1951).

Three-dimensional interaction surfaces are presented for the computation of buckling stresses for an infinitely long flat plate subjected to combinations of bending, shear, and transverse compression in its plane, a loading which approximates that occurring in the shear web of an airplane wing. The interaction surfaces are presented for two sets of boundary conditions, namely, both edges simply supported, and lower edge simply supported, upper edge clamped.

The theoretical solution, based on minimizing the potential energy is more general, treating the case in which the lower edge is simply supported and the upper edge is elastically restrained against rotation. The deflection function used satisfies, term by term, the conditions of zero deflection and zero moment at the lower edge and zero deflection at the upper edge. Lagrangian multipliers are used to satisfy the rotational boundary condition at the upper edge. A system of homogeneous simultaneous equations containing the unknown deflection coefficients is obtained. Equating the determinant of these equations to zero gives the buckling criterion in the form of a determinantal equation. In order to determine the nontrivial combinations of buckling stresses which satisfy the equations, values of two of the stress coefficients and the buckle length ratio are substituted into the determinant of coefficients and the value of the third stress coefficient which satisfies the determinantal equation is obtained. A graphical minimization of the third stress coefficient with respect to the buckle length ratio gives the combination of minimum-buckling-stress coefficients.

S. Levy (Washington, D. C.).

Yen, Kuo Tai, Salerno, V. L., and Hoff, N. J. Buckling of rectangular sandwich plates subjected to edgewise compression with loaded edges simply supported and unloaded edges clamped. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2556**, 66 pp. (1952).

The compressive stress for buckling is obtained by integrating appropriate differential equations [N. J. Hoff, same Tech. Notes, no. **2225** (1950); these Rev. **12**, 561] by the

well known Galerkin method and by a method due to D. M. A. Leggett [Proc. Roy. Soc. London. Ser. A. **162**, 62–83 (1937)]. Approximations to the buckling stress obtained by the Galerkin method approach the true buckling stress from above while those obtained by the Leggett method approach it from below. The true buckling stress is estimated as the mean of the approximate values obtained by the two methods. Numerical results for the practical range of sandwich construction are presented in a family of curves. *H. W. March* (Madison, Wis.).

Handelman, G. H. Shear center for thin-walled open sections beyond the elastic limit. J. Aeronaut. Sci. **18**, 749–754, 766 (1951).

The bending stresses are predominant in the long beams considered, and longitudinal stresses are therefore determined by the usual theory for plastic beams, neglecting the influence of shear on the plastic flow in tension and compression. The shear distribution is then determined from two equilibrium equations. Neglect of the direct stresses in the tangent plane of the section and normal to the beam axis demands the shear stress distribution to be the same at all sections for equilibrium. The second equilibrium equation then requires the longitudinal stress distributions to be an invariant curve scaled in proportion to the bending moment, in contradiction with the direct determination mentioned above in the case of plastic flow. The constant shear distribution along the beam, and associated proportional longitudinal stress distribution are used to obtain the shear centre: the point of application of the force for which the shear torque is balanced at the load considered. It varies with load, but is the same at all sections.

The reviewer believes that a varying shear distribution based on the directly computed longitudinal stress distribution should be used, in place of that determined from shear stress equilibrium, so that the shear centre would vary along the beam, leading to appreciably different results. *E. H. Lee* (Providence, R. I.).

Drucker, D. C., Prager, W., and Greenberg, H. J. Extended limit design theorems for continuous media. Quart. Appl. Math. **9**, 381–389 (1952).

The authors generalize earlier results [Proc. Amer. Soc. Civil Eng. **77**, no. 59 (1951)] to the treatment of the collapse of a perfectly plastic structure subject to an arbitrarily prescribed history of loading. Collapse is defined here as a state under which plastic flow would occur under constant loads if the accompanying change in the geometry of the structures were disregarded. Boundary conditions are assumed to be of the stress type. Admissible states of stress are then defined in such a way that the existence of a statically admissible state of stress at each stage of the loading implies and is implied by absence of collapse, whereas the existence of a kinematically admissible collapse state at any stage implies impending or previous collapse. *F. B. Hildebrand* (Cambridge, Mass.).

Pieruschka, E. Stoffgesetze und Wellen zähelastischer, isotroper Medien. Ing.-Arch. **19**, 271–281 (1951).

After long preliminaries the author discusses a material in which a linear combination of stress and stress-rate equals a linear combination of strain-rate plus rate of strain-rate. The author's theory is typical of the superposition theories discussed e.g. by Weissenberg [Abh. Preuss. Akad. Wiss. Phys.-Math. Kl. **1931**, no. 2] and von Mises [Proceedings of the 3rd International Congress for Applied Mechanics, Stockholm, 1930, Norstedt, Stockholm, 1931, pp. 3–13]. The author discusses infinitesimal harmonic oscillations. He indicates that in the usual viscous fluid or in the Maxwellian material shear waves are not observable. *C. Truesdell* (Bloomington, Ind.).

Bishop, J. F. W., and Hill, R. A theoretical derivation of the plastic properties of a polycrystalline face-centred metal. Philos. Mag. (7) **42**, 1298–1307 (1951).

In einer früheren Arbeit [Philos. Mag. (7) **42**, 414–427 (1951); diese Rev. **12**, 883] haben die Verfasser einige allgemeine Regeln bezüglich der plastischen Deformation von Kristallaggregaten unter der Annahme hergeleitet, dass sich die einzelnen Kristallite durch Gleitung entsprechend der Schmidtschen Regel deformieren. In der vorliegenden Arbeit wird die Vermutung begründet, dass die bei der plastischen Deformation eines polykristallinen Stoffes geleistete Arbeit die selbe ist, wie die welche auftreten müsste, wenn sich die einzelnen Kristallite frei deformieren würden. Mit Hilfe dieser Annahme und des Prinzips der maximalen plastischen Arbeit wird dann die Deformationsfunktion für ein Aggregat von flächenzentrierten kubischen Kristallen berechnet. Die erhaltene theoretische Kurve liegt zwischen denen von v. Mises und Tresca, die sich auf Al und Cu beziehenden experimentellen Daten dagegen zwischen den Kurven von v. Mises und der der Verfasser. Diese noch vorhandene kleine Diskrepanz rührt wahrscheinlich von einigen nicht ganz genauen theoretischen Annahmen, von der ungleichmässigen Verfestigung und anderen Deformationen als die Gleitungen entlang der Oktaederflächen her. *T. Neugebauer* (Budapest).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Wagner, H. Zur mathematischen Behandlung von Spiegelungen. Optik **8**, 456–472 (1951).

In a clear exposition, in which all details are explained in an adequate notation, the author deals with the calculation by means of quaternions of the image formed by reflection in a system of plane mirrors. The principal formula is as follows. If an object point with position vector $\mathfrak{p}_1$ (regarded as a quaternion) is reflected in a system of plane mirrors which are described by the position vectors $e_\mu \mathfrak{o}_\mu$ ($\mu = 1, 2, \cdots, \kappa$) of the feet of the perpendiculars drawn to them from the origin ($\mathfrak{o}_\mu$ being unit vectors), then the position vector of the image is

$$\mathfrak{p}_\kappa' = (-1)^\kappa (\mathfrak{Q}^{-1} \mathfrak{p}_1 \mathfrak{Q} + 2\mathfrak{r}),$$

where $\mathfrak{Q}$ is the quaternion defined by $\mathfrak{Q} = \mathfrak{Q}_1$ and the recurrence formulae $\mathfrak{Q}_\mu = \mathfrak{o}_\mu \mathfrak{Q}_{\mu+1}$, $\mathfrak{Q}_{\kappa+1} = 1$, and $\mathfrak{r}$ is a vector independent of $\mathfrak{p}_1$, namely

$$\mathfrak{r} = \sum_{\mu=1}^{\kappa} (-1)^\mu e_\mu \mathfrak{Q}_{\mu+1}^{-1} \mathfrak{Q}_\mu = \sum_{\mu=1}^{\kappa} (-1)^\mu e_\mu (\mathfrak{o}_\mu + 2\mathfrak{q}_\mu \times \mathfrak{q}_{\mu+1}),$$

where $\mathfrak{q}_\mu$ is the vector part of $\mathfrak{Q}_\mu$. Here and elsewhere the author combines the Gibbs vector notation with quaternions when convenient. The effect on the image of given motions of the object point or the mirrors is discussed. The treatment

appears to be more complete than that given by L. B. Tuckerman [Quart. Appl. Math. 5, 133–148 (1947); these Rev. 9, 549]; the author gives no reference to this or other literature. *J. L. Synge* (Dublin).

***Gorn, Saul. Rays in isotropic, nonhomogeneous media.** AF Tech. Rep. No. 6262, United States Air Force, Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio, 1951. iv+31 pp.

The author develops the geometry of light rays in inhomogeneous isotropic media, taking into general account stratified media, i.e., media in which the refractive index is constant in either parallel or concentric layers. The mathematical treatment is very elegant and clear. The reviewer would, however, like to draw the attention of the author to a paper by the reviewer [Z. Instrumentenkunde 53, 436–443 (1933)]. *M. Herzberger* (Rochester, N. Y.).

Linfoot, E. H. Error balancing in fast Schmidt cameras. Monthly Not. Roy. Astr. Soc. 111, 75–93 (1951).

The performance of a Schmidt system (consisting of a reflecting sphere and a near-flat aspheric refracting surface) may be judged by the notion of mean square image radius over the whole field. Using fifth order approximations the best field surface for monochromatic light is shown to be spherical and the best form for the aspheric plate is derived. For use over an entire spectrum it is shown that, for a given system, the curvature of the best field surface is independent of wavelength. The optimum plate profile is derived for this case and comparison is made between the performance of the classical and the modified Schmidt systems.

E. Marchand (Rochester, N. Y.).

Sturrock, P. A. Perturbation characteristic functions and their application to electron optics. Proc. Roy. Soc. London. Ser. A. 210, 269–289 (1951).

The perturbation concept consists of regarding the integrand in the variation integral as being a function of one or more "perturbation parameters." Variation of these parameters may be used to indicate chromatic variation and also geometrical aberrations, and the corresponding perturbations of Hamilton's characteristic functions are found. The theory of "first-order" perturbations is presented and leads to a pair of characteristic functions whose arguments may be, among other choices, the coordinates in the object and aperture surfaces; from these one may obtain the perturbation of the ray throughout its length. The second-order theory is also considered. Formulas are established for the geometrical, chromatic and mixed aberrations of two classes of systems; one contains all systems of rotational symmetry, the other all systems with curved axes. Extension of the theory to second and higher orders is straightforward, which apparently is not the case in the perturbation theory as developed by Schwarzschild. *E. Marchand.*

Franke, H. W. Richtungsdoppelfokussierung geschwindigkeits- und massenabweichender Teilchen in rotationssymmetrischen elektrischmagnetischen Feldern. Österreich. Ing.-Arch. 5, 371–387 (1951).

Viene estesa la teoria di Glaser [Österreich. Ing.-Arch. 4, 354–362 (1950); questi Rev. 12, 655], valevole per campi elettromagnetici a simmetria di rotazione qualsiasi, al caso di dispersione della velocità o della massa. Riferendosi alle coordinate cilindriche r, z, ψ s'introduce la funzione potenziale $Q=e\varphi/m_0-\frac{1}{2}r^2\dot{\psi}^2$ dove φ è il potenziale elettrostatico. Allora le equazioni di moto di una particella sono $\ddot{r}=Q_r$ e $\ddot{z}=Q_z$. Se Δm e $\Delta\omega$ indicano rispettivamente la dispersione della massa e della velocità angolare, le coordinate r e z del punto di focheggiatura sono espresse da

$$r=r_0+2\frac{A'}{D}\Delta m+2\frac{A}{D}\Delta\omega,\quad z=z_0+2\frac{B'}{D}\Delta m+2\frac{B}{D}\Delta\omega$$

essendo i coefficienti $A=Q_{rz}Q_{z\omega}-Q_{zz}Q_{r\omega}$, $B=Q_{rz}Q_{r\omega}-Q_{rr}Q_{z\omega}$, $D=Q_{rr}Q_{zz}-Q^2_{rz}$, $A'=Q_{rz}Q_{zm}-Q_{zz}Q_{rm}$, $B'=Q_{zr}Q_{rm}-Q_{rr}Q_{zm}$. L'autore calcola poi l'inclinazione più opportuna da dare allo schermo rispetto alla traiettoria circolare principale e le aberrazioni di apertura fino al secondo ordine.

G. Toraldo di Francia (Firenze).

Zernov, N. V. On the diffraction of plane-cylindrical electromagnetic waves. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 21, 1066–1075 (1951). (Russian)

Il problema è quello di una guida d'onda cilindrica circolare a pareti metalliche, terminata da una flangia piana indefinita. Si considerano i casi delle onde incidenti di tipo elettrico e di quelle di tipo magnetico. Lo spazio interessato dal campo viene diviso in tre regioni, una interna alla guida d'onda, una sul prolungamento di essa e una all'esterno di questo prolungamento. Le soluzioni nelle tre regioni vengono espresse come serie di funzioni cilindriche. Le condizioni di continuità danno allora per i coefficienti un sistema di infinite equazioni lineari con determinante assolutamente convergente. L'autore dà un esempio numerico di soluzione approssimata. *G. Toraldo di Francia* (Firenze).

Magnus, Wilhelm. Infinite matrices associated with diffraction by an aperture. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. EM-32, ii+20 pp. (1951).

The author summarizes his paper as follows. "Levine and Schwinger [Physical Rev. (2) 74, 958–974 (1948); 75, 1608–1609 (1949); these Rev. 10, 221, 764] reduced the problem of the diffraction of a plane scalar wave by a circular aperture [normal incidence] to the solution of an infinite system of linear equations for certain unknowns D_m which determine the field in the aperture. The D_m are power series in $2\pi a/\lambda$, where a is the radius of the aperture and λ is the wavelength. It is shown that, by solving the first l equations for the first l unknowns, the exact values of the first l coefficients of the power series for the first l unknowns D_m are obtained. Explicit recurrence formulas are given. It is shown that the solution is unique and convergent for sufficiently small values of $2\pi a/\lambda$. The mathematical nature of the system of linear equations is investigated by showing that it is connected with a problem of moments for a finite interval. It is proved that in a limiting case $[2\pi a/\lambda\rightarrow\infty]$ the transmission coefficient can still be computed although the asymptotic form of the system has no solution at all for the D_m." *C. J. Bouwkamp* (Eindhoven).

Skavlem, Steingrim. On the diffraction of scalar plane waves by a slit of infinite length. Arch. Math. Naturvid. 51, no. 4, 61–80 (1951).

The author considers the diffraction of a plane monochromatic sound wave by an infinite slit in an infinite plane screen for the case of normal incidence. He uses a somewhat unconventional, algebraic, form of elliptic-cylinder coordinates ξ, η defined by $x^2=(1+\xi^2)(1-\eta^2)$, $z=\xi\eta$, where the z-axis is perpendicular to the screen ($\eta=0$ on the screen, $\xi=0$ in the slit). In these coordinates the two-dimensional

wave equation separates into

$$(1)\qquad (1+\xi^2)u''+\xi u'+(k^2\xi^2-\lambda)u=0,$$
$$(2)\qquad (1-\eta^2)v''-\eta v'+(k^2\eta^2+\lambda)v=0.$$

The eigenfunctions $v_n(\eta)$ belonging to the eigenvalues λ_n of (2) are expanded in Tschebyscheff polynomials

$$v_n=\sum c_m T_m(\eta).$$

Corresponding solutions of (1), representing outgoing waves, are defined by

$$u_n(\xi)=i^{-n}\frac{2}{\pi}\int_{1+i\infty}^{1} e^{ik\xi\eta}v_n(\eta)\frac{d\eta}{\sqrt{1-\eta^2}}\qquad(\xi\geqq 0).$$

The diffracted field ψ is expanded as follows:

$$\psi=\sum C_n u_n(\xi)v_n(\eta),$$

where the summation is extended over odd (even) values of n if $\psi=0$ ($\partial\psi/\partial\nu=0$) on the screen. The coefficients C_n are found from the boundary conditions at $\xi=0$. The transmission coefficient of the slit is calculated and plotted as a function of k ($0\leqq k\leqq 10$). The individual contributions of the partial modes are tabulated (five decimals). Also tabulated are the first few eigenvalues λ_n and the corresponding coefficients c_m of v_n (7 or 8D) for various values of k ($\leqq 10$).

Reviewer's remark. The author is apparently unaware of earlier work of Sieger [Ann. Physik (4) **27**(332), 626–664 (1908)], Strutt [Z. Physik **69**, 597–617 (1931)], Morse and Rubenstein [Physical Rev. (2) **54**, 895–898 (1938)]. Even more remarkable, he seems to have never heard of Mathieu functions, of which extensive numerical material has been published recently [Tables relating to Mathieu functions . . . , Columbia Univ. Press, New York, 1951; these Rev. **12**, 859]. From these tables it appears that the author's values of λ_{2n+1} for $k=9$ are completely wrong; for other values of k errors of up to 15 units of the last decimal occur. The connection between the author's notation and that of the tables cited above is as follows:

$$v_{2n}(\eta)\propto \mathrm{Se}_{2n}(s,\arcsin\eta);\quad v_{2n+1}(\eta)\propto \mathrm{So}_{2n+1}(s,\arcsin\eta);$$
$$\lambda_{2n}=\mathrm{be}_{2n}(s)-s;\quad \lambda_{2n+1}=\mathrm{bo}_{2n+1}(s)-s;\quad s=k^2.$$

C. J. Bouwkamp (Eindhoven).

Brehovskih, L. M. Diffraction of electromagnetic waves from an uneven surface. Doklady Akad. Nauk SSSR (N.S.) **81**, 1023–1026 (1951). (Russian)

This paper extends to the electromagnetic case the method of an earlier paper on sound diffraction [same Doklady **79**, 585–588 (1951); these Rev. **13**; 599]. The author confines himself to the case of a perfectly conducting surface. In a foot note he corrects some minor errors in the previous paper.

F. V. Atkinson (Ibadan).

Rice, Stephen O. Reflection of electromagnetic waves from slightly rough surfaces. Comm. Pure Appl. Math. **4**, 351–378 (1951).

The author is concerned with the reflection of plane electromagnetic waves from a surface $z=f(x,y)$ which is "almost" flat. It is assumed that the small deviations of this surface from flatness are of a random nature. It is remarked that actual physical models imply "rougher" surfaces than these, but this picture has the virtue of being a simple one which shows the effect of roughness. The reflected field is determined in a fashion similar to that used by Rayleigh in the acoustics of rough walls [The theory of sound, vol. II, Macmillan, London, 1926, pp. 89–96]. These results are approximate in view of the fact that the boundary conditions are satisfied to within terms $O(f^3)$. Connected with these problems are those of surface wave propagation and electric properties of reflecting media.

A. E. Heins.

Copson, E. T. The transport of discontinuities in an electromagnetic field. Comm. Pure Appl. Math. **4**, 427–433 (1951).

This note is concerned with the propagation of discontinuities in an electromagnetic field. It is assumed that the medium is isotropic with variable dielectric and permeability and that at a point of continuity, the Maxwell equations are satisfied. If the electric and magnetic vectors suffer discontinuities across a moving surface $\psi(x,y,z)=ct$, these vector discontinuities are given by relations involving the dielectric, permeability and the gradient of ψ. For consistency of these last equations ψ satisfies the Hamilton partial differential equation for the characteristic function in a medium of given index of refraction. It has been demonstrated by Luneberg that the discontinuities in the electric and magnetic vector satisfy certain first order differential equations but his proof is involved. Following a lead of S. C. Lowell [same journal **2**, 245–291 (1949); these Rev. **11**, 279] who discussed a similar question for a scalar problem in hydrodynamics, a similar method is used for the electromagnetic problem.

A. E. Heins (Pittsburgh, Pa.).

Bremmer, H. The jumps of discontinuous solutions of the wave equation. Comm. Pure Appl. Math. **4**, 419–426 (1951).

The author studies the connection that exists between "steady-state solutions" and "pulse solutions" of Maxwell's equations in the sense of Luneberg [New York Univ. Math. Res. Group, Res. Rep. no. EM-14 (1949); these Rev. **11**, 630]. Luneberg's theory is based on the integration of Maxwell's equations over four-dimensional space-time domains. In the paper under review Luneberg's theory is simplified by introduction of the unit and delta functions of Heaviside and Dirac respectively. Equations for the eiconal function and for discontinuities on wave fronts in scalar wave propagation are treated in detail.

C. J. Bouwkamp.

Avazašvili, D. Z. On the first boundary problem of electrodynamics for a half-space. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 618–620 (1951). (Russian)

Semplice estensione delle formule di Luneberg [Mathematical theory of optics, Brown Univ., Providence, 1944; questi Rev. **6**, 107] al caso che sia presente una corrente impressa.

G. Toraldo di Francia (Firenze).

Schelkunoff, S. A. General theory of symmetric biconical antennas. J. Appl. Phys. **22**, 1330–1332 (1951).

The author gives a concise mathematical formulation of the antenna problem for symmetric biconical antennas of arbitrary angle 2ψ. He shows that the input admittance may be derived from a terminating admittance Y_t which is expressed as an infinite series.

$$KY_t=-\frac{2\eta}{V(l)}\sum_{k=1}^{\infty} b_{2k+1}P_{2k+1}(\cos\psi),$$

where $\eta=(\mu/\epsilon)^{1/2}$, $K=(\eta/\pi)\log\cot\frac{1}{2}\psi$ and the b's are solutions of an infinite set of algebraic equations. These may be solved by successive approximations, and it is shown that the first approximation leads to a solution for the input admittance which approaches the exact expression for the

two limiting cases $\psi=0°$, $90°$. Thus it is conjectured that this term represents a good approximation for all values of ψ.

M. C. Gray (Murray Hill, N. J.).

Levin, M. L. On the derivation of the fundamental equation of the theory of slotted antennas. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 787–794 (1951). (Russian)

Viene discusso il fatto che il problema dell'antenna a fenditura dipende da un'equazione integrodifferenziale analoga a quella che regge il caso delle antenne metalliche sottili. E' un principio di complementarità, di cui è evidente la relazione con quello di Babinet.

G. Toraldo di Francia (Firenze).

Levin, M. L. On the theory of slotted antennas in a circular wave guide. Akad. Nauk SSSR. Žurnal Tehn. Fiz. **21**, 772–786 (1951). (Russian)

Vengono ripresi gli studi di Pistolkors [stesso Žurnal **17**, 365–376, 377–388 (1947)] sulla radiazione da parte di fenditure praticate nelle pareti di una guida d'onda cilindrica circolare. Per fenditure longitudinali la forza elettrica trasversale sulla superficie cilindrica viene sviluppata in serie di Fourier

$$E_\varphi=\sum a_n f_n(z)\cos(n\varphi+d_n).$$

Quindi viene studiata la radiazione da parte di ciascuna armonica. Si può così facilmente valutare la radiazione totale e la conduttanza di radiazione dell'antenna. Quando il raggio del cilindro è piccolo rispetto alla lunghezza d'onda, la conduttanza di radiazione di una fessura è metà di quella che si ha per un raggio infinito. Si studia poi il regime all'interno della guida, per dedurne l'adattamento di impedenza con lo spazio esterno. Viene anche esaminato il caso delle fessure trasversali; a questo proposito l'autore rileva un errore nel quale era incorso Pistolkors, che aveva fatto uso del solo potenziale vettore elettrico, mentre per ottenere $E_\varphi=0$ sul cilindro bisogna anche considerare un potenziale vettore magnetico.

G. Toraldo di Francia (Firenze).

Svešnikov, A. G. The principle of limiting absorption for a wave-guide. Doklady Akad. Nauk SSSR (N.S.) **80**, 345–347 (1951). (Russian)

Si vuole una soluzione dell'equazione $\Delta u+k^2u=-f(z, M)$, con $M=(x, y)$, all'interno di un cilindro qualsiasi, con la condizione $u=0$ o $\partial u/\partial n=0$ sulla superficie laterale; si vuole escludere la presenza di onde che vengano dall'infinito. Si può allora sostituire l'usuale condizione di radiazione con quella dell'assorbimento limite. Si cerca cioè una soluzione con numero d'onde $k_1^2=k^2+i\epsilon$ e quindi si fa tendere ϵ a zero. L'autore dimostra la rigorosità del procedimento.

G. Toraldo di Francia (Firenze).

Jessel, Maurice. Rayonnement d'une antenne placée dans un guide d'onde à section rectangulaire. C. R. Acad. Sci. Paris **233**, 783–785 (1951).

The known radiation field of an antenna in free space may be used to determine its radiation field inside a rectangular wave guide by the method of images. The images of the antenna in the plane faces of the guide form a plane net, whose basic elements are: A, the element itself; B, the image of A in the xy-plane; C, the image of A in the yz-plane; and D, the image of B in the yz-plane. For each propagation mode in the guide the field is expressed as a sum of the fields from the four basic sources. The resulting formulas also represent the field from an arbitrarily oriented slot in one of the guide walls, and the method is applied to a slotted guide for the H_{10} mode.

M. C. Gray.

Lurye, Jerome Robert. Electromagnetic scattering matrices of stratified anisotropic media. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-31**, iii+46 pp. (1951).

Schwinger's integral-equation-variational technique is applied to the problem of the scattering of monochromatic plane electromagnetic waves by a continuously stratified anisotropic layer. Variational formulae for the reflection and transmission matrices are obtained by using the theory of linear spaces.

To give some account of the method briefly, we take the special case when the waves are incident normally on an isotropic non-homogeneous layer. Waves for which $\mathbf{E}_0=\mathbf{i}e^{ikz-i\omega t}$ are incident on the layer $0\leqq z\leqq a$ in which the continuous complex dielectric constant is $\epsilon(z)$; outside the layer $\epsilon(z)=1$. The electric force is everywhere parallel to the axis of x and satisfies the differential equation $E''+k^2\epsilon E=0$. In $z<0$, $E=e^{ikz}+Re^{-ikz}$; if $z>a$, $E=e^{ikz}+Se^{ikz}$. The problem is to find R and S. If we write the differential equation as $E''+k^2E=-BE$, where $B(z)=k^2\{\epsilon(z)-1\}$, Green's function theory leads to an integral-equation for E, viz.

$$E(z)=e^{ikz}+\tfrac{1}{2}ik\int_0^a e^{ik|z-z'|}B(z')E(z')dz'$$

for all z. And hence

$$R=\tfrac{1}{2}ik\int_0^a e^{ikz}B(z)E(z)dz,\quad S=\tfrac{1}{2}ik\int_0^a e^{-ikz}B(z)E(z)dz.$$

An equivalent formula for R is

$$R=\tfrac{1}{2}ik\frac{\left\{\int_0^a e^{ikz}B(z)E(z)dz\right\}^2}{\int_0^a BE^2dz-\tfrac{1}{2}ik\int\int_0^a e^{ik|z-z'|}B(z)E(z)B(z')E(z')dz'dz}$$

and this expression has the property of being stationary for small variations of E about the value given by the integral equation. By inserting a suitably chosen trial value of E, the formula gives a good approximation to R.

To deal with S, we introduce a second integral equation

$$F(z)=e^{-ikz}+\tfrac{1}{2}ik\int_0^a e^{ik|z-z'|}B(z')F(z')dz'$$

so that $F(z)$ is the electric force when the incident force is $E_0e^{-ikz-i\omega t}$. We then have an alternative formula for S, viz.

$$S=\tfrac{1}{2}ik\int_0^a e^{ikz}B(z)F(z)dz.$$

The expression for S which is stationary for small variations of E and F is

$$S=\tfrac{1}{2}ik\frac{\int_0^a e^{ikz}BFdz\cdot\int_0^a e^{-ikz}BEdz}{\int_0^a BEFdz-\tfrac{1}{2}ik\int\int_0^a e^{ik|z-z'|}B(z)E(z)B(z')F(z')dz'dz}.$$

The anisotropic case is more complicated since $\epsilon(z)$ is a matrix function of z, and vector solutions have to be considered. But the special case does indicate many of the ideas involved.

E. T. Copson (St. Andrews).

Eckart, Gottfried. Le rayonnement d'un dipôle magnétique dans un milieu stratifié de symétrie sphérique. Ann. Télécommun. **5**, 173–178 (1950).

The author investigates the radiation of a radial magnetic Hertzian dipole immersed in a spherically stratified medium

of dielectric constant $\epsilon=\alpha+\beta/r$. The field is expressed in terms of Debye's magnetic potential, which is a solution G of the wave equation. By separation in spherical coordinates, two forms of Green's function G are derived which differ in respect to normalization. The first form, suggested by Meixner [Math. Z. 36, 677–707 (1933); Ann. Physik (5) 29(421) 97–116 (1937), pp. 111–112] is a representation by the discrete and the continuous spectra of eigenfunctions [Legendre, Laguerre-Whittaker functions] of the wave equation. The second form is suggested by known results for $\beta=0$ (homogeneous medium). Comparison of the two forms leads to a certain relation between various Bessel and Whittaker functions of special arguments. There are many misprints in the paper. *C. J. Bouwkamp* (Eindhoven).

Wait, James R. The magnetic dipole over the horizontally stratified earth. Canadian J. Physics 29, 577–592 (1951).

The behavior of a small current-carrying wire loop over a horizontally stratified earth is investigated. The layers are considered to have a contrast in conductivity and dielectric constant only. Both harmonic steady-state and step-function current sources are considered. (Author's summary.)

C. J. Bouwkamp (Eindhoven).

Kaden, Heinrich. Eine allgemeine Theorie des Wendelleiters. Arch. Elektr. Übertragung 5, 534–538 (1951).

The author uses the work of J. R. Pierce [Proc. I. R. E. 35, 111–123 (1947), p. 122] on the propagation of electromagnetic waves along a helix to obtain formulas and graphs for the power transmitted, the absorption and the phase velocity of the waves. *B. Friedman* (New York, N. Y.).

Bauer, H. Tensorielle Behandlung elektrotechnischer Probleme. Österreich. Ing.-Arch. 6, 4–11 (1951).

An explanatory article on the transformation theory of interconnected stationary and rotating electrical networks. The analysis of the Wheatstone bridge, the repulsion motor and an industrial control system (consisting of a group of interconnected rotating machines and transformers) serve as illustrations of the concepts of connection tensor and impedance tensor. *G. Kron* (Schenectady, N. Y.).

Smythe, W. R. The capacitance of a circular annulus. J. Appl. Phys. 22, 1499–1501 (1951).

The exact Newtonian capacity of a circular annulus has not been determined, and the present note contains a number of numerical approximations to it. The first method, which establishes definite bounds, is to start with the equilibrium distribution on a disk, and to go several steps with an alternating process in which one subtracts the induced distribution on an infinite plane conductor outside the hole (or inside the outer perimeter of the annulus) due to that part of the distribution already obtained inside the hole (or outside the outer perimeter). The bounds are very near to coincidence if the annulus is not too narrow; if the ratio of outer to inner radius is 1.5, the capacity lies between 69.44 and 69.51 times the outer radius. The second method is to apply a correction to values obtained by Higgins and Reitan [Trans. Amer. Inst. Elec. Engrs. 70, 926–933 (1951)] who used a method of dividing the annulus into sub-annuli. The correction is made by considering an infinite flat strip, where the exact solution is known, computing the error in Higgins' and Reitan's method for this case, and applying this correction in the present case. The third method, applicable only to very narrow annuli, amounts to approximating the equilibrium potential by the potential along the center circle of the annulus of a circular wire displaced from the center circle perpendicular to its plane a distance which gives the correct result in the case of the infinite strip.

J. W. Green (Princeton, N. J.).

✓ ***Darwin, Charles G. The refractive index of an ionised gas.** Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 1, pp. 593–600. Amer. Math. Soc., Providence, R. I., 1952.

L'autore ricorda la difficoltà che si incontra nel calcolo dell'indice di rifrazione di un mezzo ionizzato. Se prima si passa al limite di una distribuzione continua per le cariche positive e poi si passa al limite di uno spostamento infinitesimo per l'elettrone, si trova una formula del tipo di Sellmeyer (cioè di quella che nei dielettrici corrisponde al non tener conto del campo interno). Se invece si effettuano i passaggi al limite in ordine inverso, si ottiene una formula del tipo di Lorentz (quale nei dielettrici nasce dalla considerazione del campo interno). Ma l'autore osserva che in questo secondo caso si esclude che un elettrone possa trovarsi molto vicino a un protone. Se invece si tiene conto di questa possibilità, si arriva a dimostrare la validità della formula del tipo di Sellmeyer. I relativi sviluppi matematici si trovano in due lavori precedenti dello stesso autore [Proc. Roy. Soc. London. Ser. A. 146, 17–46 (1934); 182, 152–166 (1943)]. *G. Toraldo di Francia* (Firenze).

Quantum Mechanics

Schönberg, M. Physical applications of the resolvent operators. I. On the mathematical formalism of Feynman's theory of the positron. Nuovo Cimento (9) 8, 651–682 (1951).

The author gives a (mathematically unrigorous) account of the theory of the resolvent $R(\lambda)=(H-\lambda I)^{-1}$ of a Hermitian operator H on a complex Hilbert space (I = identity operator) from a viewpoint adapted to the treatment of the Feynman positron theory [Physical Rev. (2) 76, 749–759 (1949)]. The theory uses such operators as

$$V_F(t)=(2\pi i)^{-1}\int_C \exp(-i\lambda t)R(\lambda)d\lambda,$$

where C is obtained from the x-axis by shifting the negative axis down and the positive axis up slightly; in terms of V_F the Feynman kernel K_+^A is $\langle x|V_F(t-t')|x'\rangle$. An infinite series for V_F is obtained by expanding $R(\lambda)$ in a geometric series around $H_0-\lambda I$, where H_0 is the free electron Hamiltonian. This is related to Feynman's expansion, which can be obtained from it by deforming C into the real axis, but in general the two expansions differ materially. The case of a time-dependent Hamiltonian is treated in a parallel fashion. *I. E. Segal* (Chicago, Ill.).

Yovanovitch, D. K. Sur le principe de l'incertitude et "La causalité de groupe" dans la physique contemporaine. Bull. Soc. Math. Phys. Serbie 3, no. 1–2, 1–10 (1951). (Serbo-Croatian. French summary)

Salam, Abdus. Overlapping divergences and the S-matrix. Physical Rev. (2) 82, 217–227 (1951).

The reviewer [same Rev. 75, 1736–1755 (1949); these Rev. 11, 145] gave a proof of the finiteness of the S-matrix after renormalization, in the quantum electrodynamics of spin-½ particles. In this proof, various mathematical ques-

tions arose which were discussed only in rather vague terms and not rigorously formulated. These questions were difficult to formulate, not because of any deep-lying obscurity, but simply because the questions concerned the analysis of extremely complicated multiple integrals for which no adequate notation existed. In the words of the author, an adequate notation is one which is "both concise, and intelligible to at least two persons of whom one may be the author". In the present paper a notation is for the first time presented which is adequate according to this definition. This is an important contribution, for it means that the whole discussion of the renormalization technique is now for the first time placed upon a clear and rigorous basis.

As a first application of his methods, the author fills the gaps in the reviewer's paper mentioned above. Then he goes on to discuss the much more complicated situations which arise in applying the renormalization method to theories of spin-zero mesons interacting with spin-$\frac{1}{2}$ particles (nucleons). He proves that the renormalization program can be carried through and leads to a divergence-free S-matrix in two cases, (i) if the mesons are charged and scalar, or (ii) if the mesons are pseudoscalar and either charged or neutral. The analysis is explained with great clarity, but is too complicated to be summarized here. *F. J. Dyson.*

Salam, Abdus. Divergent integrals in renormalizable field theories. Physical Rev. (2) **84**, 426–431 (1951).

The proofs of the success of the renormalization method in making finite the theories of scalar and pseudoscalar mesons, in interaction with the nucleon field and with the electromagnetic field, have been given previously by the author [see the preceding review]. The proofs depended on a free use of a certain subtraction procedure, by which a finite part could be extracted from a given divergent integral and separated from other infinite parts which were interpreted as renormalization effects. A general proof is here given that this subtraction procedure, when applied to a divergent integral in the S-matrix of a renormalizable field theory, does in fact lead to an absolutely convergent and unique remainder. The proof is long and complicated, because of the difficulty in devising an adequate notation to describe the integrals with which one is dealing; only straightforward and elementary algebraic manipulations are involved. With this paper the formal applicability of renormalization to meson theories is finally established. *F. J. Dyson.*

Dyson, F. J. The renormalization method in quantum electrodynamics. Proc. Roy. Soc. London. Ser. A. **207**, 395–401 (1951).

This is a nontechnical account of the main ideas behind a series of papers, of which two have appeared [see the two following reviews], dealing with the divergences in electrodynamics. A new representation called the "intermediate", of which the interaction and Heisenberg representations are each a special case, is introduced in which high-frequency fluctuations are represented in the field operators and low-frequency fluctuations in the state vector. Observables are averages over finite space-time intervals of intermediate operators. In the series of papers the author expects (i) to prove that all matrix elements of the intermediate field averages are finite after renormalization, (ii) to show that the state vector satisfies a divergence-free Schrödinger equation, (iii) to devise practical methods for the approximate solution of Schrödinger's equation even for bound states. Finally, the author states that in view of the recent paper of Salam [see the second preceding review], there seems a good prospect of using the intermediate representation to give a divergence-free treatment of pseudo-scalar meson theory. *A. J. Coleman* (Toronto, Ont.).

Dyson, F. J. Heisenberg operators in quantum electrodynamics. I. Physical Rev. (2) **82**, 428–439 (1951).

This is the first in a series of papers which will attempt to show that the use of mass and charge renormalization is sufficient to enable quantum electrodynamics to assign finite values to all measurable quantities in all situations. The author has already shown this to be the case for the S matrix [Physical Rev. (2) **75**, 486–502 (1949); these Rev. **10**, 418] but a knowledge of the S matrix does not provide knowledge of local quantities such as current density. The author regards an actually measurable quantity, for example, field-strength, as an average $F_{\mu\nu}(R)$ over a finite space-time region R of a Heisenberg point operator $F_{\mu\nu}(x)$. The whole of the present preliminary paper is devoted to the proof of an expansion theorem permitting the explicit calculation of Heisenberg point operator as a series of normal products with c-number coefficients M. The body of the paper, discusses the evaluation of the M, each of which is the sum of numbers $M(S_G)$ arising from connected graphs G. The $M(S_G)$ are defined by a combination of Abel integration and a new analytic continuation method. A generalized $M_\Gamma(S_G)$ is introduced which depends on vectors Γ_i and parameters m and λ. For sufficiently large m and λ, the condition Re $(\Gamma_{i0})>0$ ensures the existence of $M_\Gamma(S_G)$. Analytic continuation defines its value for values of m and λ of physical interest and the $M(S_G)$ results in the limit $\Gamma_{i0}\to 0$. The whole process is summarized in a set of rules analogous to the Feynman-Dyson rules for calculating the S-matrix. *A. J. Coleman* (Toronto, Ont.).

Dyson, F. J. Heisenberg operators in quantum electrodynamics. II. Physical Rev. (2) **83**, 608–627 (1951).

The program begun in the paper (C), reviewed above, is continued. However, its scope is enlarged by the intervening discovery of the intermediate representation of which a non-technical account was given in (E) [see the 2d preceding review]. The main problem of the present paper is to show that the divergences in the coefficients M, defined in (C), can all be removed by mass or charge renormalization. First, the intermediate representation is given precise mathematical definition by the introduction of a function $g(t-t')$ which corresponds to an adiabatic switching on of the interaction between the particles and the field. The coefficients M for the field-strengths in intermediate representation are then studied in detail. The discussion is made tractable by the use of Gupta's formalism [Proc. Phys. Soc. Sect. A. **64**, 50–53 (1951); these Rev. **12**, 572; see also Gardner, ibid., 426 (1951)], and though considerably more complicated than the previous treatment of the S-matrix gives rise to no essentially new problem. The author concludes that renormalization suffices to remove all divergences for the electromagnetic field operators $A_{\mu a}(p)$ and the matter-field operators $\psi_a(p)$ in intermediate representation. This conclusion continues to hold in the limit $g(t)\to 1$, that is, the Heisenberg representation. A special argument shows that though the current operator may contain divergences in intermediate representation it contains none in Heisenberg representation. Thus, point (i) of the three point program of (E) has been carried out for electrodynamics. *A. J. Coleman* (Toronto, Ont.).

Dyson, F. J. The Schrödinger equation in quantum electrodynamics. Physical Rev. (2) **83**, 1207–1216 (1951).

A detailed argument shows that the Hamiltonian in intermediate representation for quantum electrodynamics contains no explicit or implicit divergences. The proof involves the delicate and laborious methods which the author has employed so successfully in previous papers. The final section envisages the possible application of methods suggested by this paper to problems involving bound states. The outstanding difficulty is to show that the series in powers of e for the operator involved converges. The present paper shows only that each term in the series is finite. That this is no inconsiderable difficulty is indicated by a paper of Ferretti [Nuovo Cimento (9) **8**, 108–131 (1951); these Rev. **12**, 890]. *A. J. Coleman* (Toronto, Ont.).

Snyder, Hartland S. Remarks concerning the adiabatic theorem and the S-matrix. Physical Rev. (2) **83**, 1154–1159 (1951).

The purpose of this paper is to clarify the assumptions underlying the current treatments of scattering problems in quantum mechanics. Suppose a system defined by a hamiltonian $H=H_0+V$, where H_0 is the hamiltonian of the non-interacting parts of the system, and V is the interaction energy. The "adiabatic theorem" states that if the system started in the remote past in a stationary state of the non-interacting hamiltonian H_0, and if the interaction V were then switched on very gradually over a long period of time T, the state of the system after switching on V would be a stationary state of the total hamiltonian H in the limit as $T\rightarrow\infty$. This theorem is often used in the treatment of scattering problems in order to justify using an unperturbed wave-function for the initial state of the system. The author examines the validity of the theorem by constructing explicitly the solution of the equations of motion of the system when an exponentially growing factor exp (αt) multiplies the interaction V. The solutions are rather simple in form and have the correct limiting behaviour as $\alpha\rightarrow 0$. The standard formulae of scattering theory are thus derived in a rigorous way.

In a short final section, the author remarks that the adiabatic theorem as here stated is in many important cases not valid. In such cases the definitions of the scattering matrix (S-matrix) in time-independent perturbation theory and in time-dependent theory will not in general be equivalent. *F. J. Dyson* (Ithaca, N. Y.).

Utiyama, Ryôyû. On the covariant formalism of the quantum theory of fields. II. Progress Theoret. Physics **6**, 65–95 (1951).

The methods described in a previous paper by the author [same journal **5**, 437–458 (1950); these Rev. **12**, 379] are here applied to construct a new covariant formulation for quantum electrodynamics. The formalism is similar to that of Rosenfeld [Ann. Physik (5) **5**(**397**), 113–152 (1930)], but is developed in such a way as to be manifestly covariant in form. It differs from the usual formalism in that it makes the Lorentz gauge condition an operator equation instead of only a supplementary condition. Its physical content and consequences are shown to be identical with those of the usual formalism. Finally, the same methods are again applied in order to construct a new covariant formulation for vector meson theory, of the same type as that given for electrodynamics. *F. J. Dyson* (Ithaca, N. Y.).

Koba, Ziro, Mugibayashi, Nobumichi, and Nakai, Shinzô. On gauge invariance and equivalence theorems. Progress Theoret. Physics **6**, 322–341 (1951).

This discussion of the ambiguous and non-gauge-invariant results that can sometimes arise in the course of perturbation-theory calculations in quantum field theory is considerably clearer and more satisfactory than previous discussions of the same problems [Y. Katayama, same journal **5**, 272–282 (1950); H. Fukuda and T. Kinoshita, ibid. **5**, 1024–1032 (1950); these Rev. **12**, 151, 890]. It is shown that in cases where non-gauge-invariant terms appear these always arise from the evaluation of integrals of the form

$$\int F(t)dt-\int F(t-k)dt, \tag{1}$$

where t is a variable and k a constant 4-vector, and the integration is over all four components of t from $(-\infty)$ to $(+\infty)$. When (1) is evaluated, it is usual to combine the two integrands before integrating over t. This will generally lead to a finite non-zero result for (1), in circumstances when either of the two integrals taken separately would be linearly or quadratically divergent. Such a non-zero value for (1), although obtained by apparently legitimate mathematical procedures, is to be discarded and replaced by zero in order to obtain physically consistent results. It is shown that in this way, using physical arguments to guide the mathematics, all the ambiguities and inconsistencies of the calculations are effectively removed. *F. J. Dyson.*

Katayama, Yasuhisa. On the positron theory of vacuum. Progress Theoret. Physics **6**, 309–321 (1951).

In previous papers [same journal **5**, 272–282, 1054–1055 (1950); these Rev. **12**, 151] the author has discussed the possible ambiguities which arise in making calculations in quantum electrodynamics, and has tried to formulate a set of rules for giving a unique value to every ambiguous integral, independent of the physical situation in which the integral may appear. The attempt to formulate such rules is a departure from the generally accepted philosophy of Tomonaga and Schwinger, who preferred to use physical arguments to assign values to quantities which were mathematically ambiguous, with the result that the same mathematical expression might in principle have different interpretations in different physical circumstances. In this paper the author studies in detail the problem of the polarization of the vacuum in an external electromagnetic field. He finds in this problem various ambiguous and non-gauge-invariant terms. The rules required to ensure the vanishing of these terms lead to difficulties when applied to more complicated problems in meson theory. He concludes that a completely satisfactory set of rules of interpretation of ambiguous quantities has still to be found. *F. J. Dyson.*

Rayski, J. On the quantum theory of reciprocal fields and the correspondence principle. Proc. Phys. Soc. Sect. A. **64**, 957–968 (1951).

The ideas of Born and Yukawa of non-localizability and reciprocity are very attractive, according to the author, but have not yet been sufficiently developed. "The decisive problems which need solution are: (i) how to interpret non-local fields; (ii) how to take account of interaction between non-local fields; (iii) how the non-local fields should be quantized; (iv) whether it is possible to secure a reciprocity of interaction; (v) whether it is possible to secure correspondence with local field theory."

The present paper throws light on all these questions in the case of a non-local complex scalar field in scalar interaction with a local real scalar field. A generalization of the Δ-function is introduced and used to formulate the field equations in the form of an integral equation which is not reducible to an equivalent differential equation. Quantization is effected by assuming free-field commutation relations for incoming and outgoing waves, whence commutation relations for the perturbed wave functions can be computed by iteration. At least in their most immediate meaning, it appears impossible to reconcile desiderata (iv) and (v).
A. J. Coleman (Toronto, Ont.).

Moshinsky, Marcos. Boundary conditions and time-dependent states. Physical Rev. (2) **84**, 525–533 (1951).

Using a phenomenological description of a nuclear reaction in terms of boundary conditions and a representation of the state of the system in terms of two wave functions (Fock space) the temporal course of a resonant scattering and a compound nucleus disintegration is considered. For a suitably simplified model the complete solution is obtained using a generalized Hankel transform. The dependence of the results on the poles of the scattering matrix is discussed.
K. M. Case (Ann Arbor, Mich.).

Moshinsky, Marcos. Quantum mechanics in Fock space. Physical Rev. (2) **84**, 533–540 (1951).

The discussion in the paper reviewed above of resonant reactions using a Fock space is translated into quantum mechanical language. A Schrödinger equation is written using a Hamiltonian determined so as to be the generator of infinitesimal translations in time. Transformation theory is developed. From this it follows that the generalized Hankel transform of the previous paper is an example of the expansion theorem in the Fock space. An explicit expression is given for the unitary matrix relating the state of the system at two different times. *K. M. Case.*

Wessel, Walter. On relativistic quantum mechanics and the mass operator. Physical Rev. (2) **83**, 1031–1037 (1951).

This paper discusses the consequences of a theory of the electron developed previously by the author [Z. Naturforschung **1**, 622–636 (1946); these Rev. **9**, 128] and by Bopp [ibid. **3a**, 564–573 (1948); these Rev. **10**, 663]. [See also W. Wessel, Physical Rev. **76**, 1512–1519 (1949); these Rev. **11**, 318.] The essential feature of this theory is that the equation of motion of the electron including radiative reactions is first set up in the classical theory, and quantization is afterwards applied to the equation of motion so obtained. The electron then has an infinity of internal degrees of freedom, corresponding to the degrees of freedom of the external Maxwell field in the usual quantum electrodynamics.

The author shows here how the operators describing the electron constitute an infinite-dimensional representation of the Lorentz group, of a type not hitherto known. Only recently have similar, but not identical, representations been discussed in the literature [J. Gelfand and A. Yaglom, Doklady Akad. Nauk SSSR **59**, 655–658 (1948); these Rev. **9**, 496]. The "mass operator" is defined generally as the invariant scalar appearing in the characteristic equation of the representation; e.g. in the case of the Dirac spinor representation the mass operator reduces to a pure number, the mass which appears in the Dirac equation. In the author's theory the mass operator has a much more general form, and in general possesses an infinity of discrete eigenvalues. *F. J. Dyson* (Ithaca, N. Y.).

Coester, F. Quantum electrodynamics with nonvanishing photon mass. Physical Rev. (2) **83**, 798–800 (1951).

A vector field satisfying the Klein-Gordon equation and the commutation rules of the Maxwell field is expressed as the sum of a vector part obeying the commutation rules of the Proca field and a part which is the gradient of a scalar field $B(x)$. Scalar photons are excluded by imposing the conditions $B^{(-)}(x)\Psi=0$, $B^{(+)}(x)\Psi=0$ by means of the negative and positive frequency parts of the operator $B(x)$. In the interaction of the resulting field with a current-charge-density vector a theory is derived which passes continuously into quantum electrodynamics for vanishing mass of the associated quanta. The theory is not gauge-invariant.
C. Strachan (Aberdeen).

Rosen, Nathan. Particle spin and rotation. Physical Rev. (2) **82**, 621–624 (1951).

A particle is considered as a small sphere whose rotation is described by Euler parameters. In a space of co-ordinates related to these parameters angular momentum operators are derived, the standard commutation rules are applied, and a Schrödinger equation is formulated. About a diameter fixed in space the angular momentum component takes $2S+1$ values ($S=0, \frac{1}{2}, 1, \cdots$) and for each such state there are $2S+1$ different states characterized by different values of the angular momentum component about a diameter fixed in the sphere. For the electron there are thus $(2\times\frac{1}{2}+1)^2=4$ states. The effect of an external magnetic field and of translational motion are briefly considered. *C. Strachan.*

Širokov, Yu. M. The relativistic theory of spin. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 748–760 (1951). (Russian)

The author sets up relativistically invariant definitions of spin and center of mass for a system of particles and obtains a covariant separation of spin and orbital angular momentum. He extends these ideas to the case of a particle with spin, described quantum-mechanically by a relativistically invariant system of first-order equations. The invariant square of the spin is found to be quantized at the values $l(l+1)h^2/4\pi^2$ where l is an integer or half-integer. The relativistic definition of center of mass leads to the "Zitterbewegung" of the Dirac electron. In the case of a system with continuous internal degrees of freedom it is found that there exists a spectrum of masses that increase with increasing spin. *N. Rosen* (Chapel Hill, N. C.).

Trkal, V. General Lorentz transformation of Dirac's wave function. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **50** (1949), 407–425 (1951).

English version of an earlier paper in Czech [Rozpravy II. Třídy České Akad. **59**, no. 32 (1949); these Rev. **12**, 658].

Mayot, Marcel. Le calcul des perturbations en mécanique quantique. C. R. Acad. Sci. Paris **234**, 920–921 (1952).

Visconti, Antoine. Sur certaines transformations fonctionnelles de l'équation d'évolution. C. R. Acad. Sci. Paris **234**, 817–819 (1952).